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Approximating the minimum triangulation of convex 3-polytopes with bounded degrees ☆

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Abstract

Finding minimum triangulations of convex 3-polytopes is NP-hard. The best approximation algorithms only give an approximation ratio of 2 for this problem, which is the best possible asymptotically when only combinatorial structures of the polytopes are considered. In this paper we improve the approximation ratio of finding minimum triangulations for some special classes of 3-dimensional convex polytopes. (1) For polytopes without 3-cycles and degree-4 vertices we achieve a tight approximation ratio of $3/2$. (2) For polytopes where all vertices have degrees at least 5, we achieve an upper bound of $2-1/12$ on the approximation ratio. (3) For polytopes with n vertices and vertex degrees bounded above by Δ we achieve an asymptotic tight ratio of $2-\Omega(1/\Delta)-\Omega(\Delta/n)$. When Δ is constant the ratio can be shown to be at most $2-2/(\Delta+1)$.

Keywords: Convex 3-polytopes; Minimum triangulation; Approximation algorithms

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1. Introduction

A *triangulation* of a d-dimensional polytope is its subdivision into a set of simplices, such that the simplices do not overlap and intersect only at common faces. We are interested in three-dimensional polytope triangulations (also called *tetrahedralizations*), which have important applications in computer graphics, finite element analysis, computer-aided design, etc. as well as having fundamental theoretical significance. In particular, we want to find triangulations consisting of a small number of tetrahedra. Since we only consider polytopes in three dimensions, we simply call '3-polytopes' as 'polytopes' in this paper.

The problem of polytope triangulation has been studied extensively. Convex 3-polytopes can always be triangulated, but triangulations of the same polytope can differ in size, i.e., contain different numbers of tetrahedra. It is shown in [2] that finding a minimum triangulation, i.e., a triangulation with the minimum possible size, is NP-hard. There are several algorithms to triangulate a polytope, but not specifically addressing the problem of minimum triangulation. For example, the simple *pulling* heuristic [8], which picks a vertex and connects it to all other non-adjacent faces of the polytope, gives an approximation ratio of 2 for finding minimum triangulations. Though simple, no better triangulation algorithms were known for a long time.

In [5] a new triangulation algorithm was given, by making use of the properties of 3-cycles. A 3-*cycle* is a cycle of length three on the surface graph of a polytope such that both sides contain some other vertices (i.e., the triangular faces of the polytope are not regarded as 3-cycles). A 3-cycle separates a polytope into two parts. The idea of the algorithm (which we call **CutPull** in this paper) is very simple: the polytope is partitioned along all the 3-cycles into subpolytopes, each is free of 3-cycles. Then the pulling heuristic is applied to each resulting subpolytope. It was shown that this algorithm gives an approximation ratio of $2 - \Omega(1/\sqrt{n})$ where n is the number of vertices of the polytope.

Although the above bound seems to be a slight improvement only, it was proved in the same paper that this approximation ratio is the best possible, for algorithms that only consider the combinatorial structure of the polytopes. This lower bound is proved by utilizing a property of vertex-edge chain structures (VECSs), first introduced in [2]. A VECS of size s consists of the vertices $(a, b, q_0, q_1, \ldots, q_{s+1})$, forming the set of triangular faces $\{aq_iq_{i+1}, bq_iq_{i+1}\ (0 \leqslant i \leqslant s)\}$ (Fig. 1(a)). It consists of a chain of degree-4 vertices. An important property of the VECS is [5]: if the graph of a polytope contains a VECS as a substructure, and the interior edge ab (called the *main diagonal*) is not present in a triangulation of the polytope, then in this triangulation at least $2s$ tetrahedra are 'incident' to the VECS. On the other hand, if ab is present, $s + 1$ incident tetrahedra may be sufficient for the triangulation. Note that in a VECS there are two vertices a and b having high degrees.

In view of these results, the following question is raised in that paper: can the approximation ratio be improved when the maximum vertex degree of the polytope is bounded? Another interesting question is whether there are special types of polytopes that have optimal triangulations or with better approximation ratios using **CutPull**. In this paper we give some results about these questions.

The rest of this paper is organized as follows:

- In Section 2, we give new bounds on the relationship between the size of minimum triangulation, the maximum vertex degree, and the number of 3-cycles of a polytope, improving the previous results given in [5].

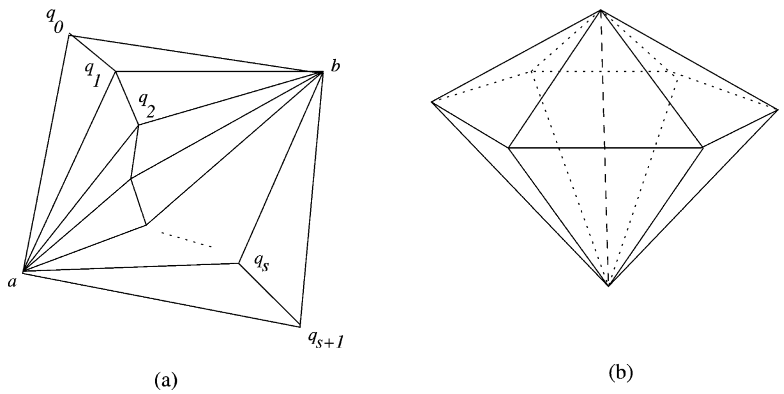

Fig. 1. (a) A VECS of size s. (b) A bipyramid with $n-2$ vertices in the middle chain; here $n=8$.

- In Section 3, we show that when a polytope has no degree-4 vertices and no 3-cycles, any triangulation of the polytope has at least $\approx 4n/3$ tetrahedra, and this bound is achievable. Thus we can prove that **CutPull** gives an improved approximation ratio, $3/2$ instead of 2, and this bound is tight. For polytopes with all vertices of degree-5 or above, an upper bound $2-1/12$ on the approximation ratio can be proved.
- In Section 4, we give a generalized analysis of the **CutPull** algorithm for polytopes with vertex degrees bounded above by Δ. The analysis gives an asymptotically tight approximation ratio for algorithms that only consider the combinatorial structures of polytopes. In particular, the ratio is better than 2 for the constant-degree case, e.g., $12/7$ for $\Delta=6$ and $7/4$ for $\Delta=7$.

2. Preliminaries

Throughout this paper, let P be a convex polytope in $\Re^3$ with n vertices, Δ be the maximum vertex degree, and k be the number of 3-cycles. We only consider polytopes with vertices in *general position*, i.e., no four vertices are coplanar. The size t of a triangulation and the number of interior edges e_i it uses satisfy the formula $t=e_i+n-3$ [1]. Let t_m be the size of minimum triangulation of P, and e_m be the number of interior edges in this minimum triangulation. It follows that $t_m=e_m+n-3$. It is also shown in [5] that e_m is related to Δ under the restriction that the polytope has no 3-cycles, by the formula $2e_m(\Delta+1)\geqslant n$. In this section we improve this formula by tightening the inequality by almost a factor of 2 (the constant-factor improvement is important when we come to Section 4), and also extending it to the case with 3-cycles.

Lemma 1. *For a polytope P with no 3-cycles and $n>4$ vertices, $e_m\Delta\geqslant n-2$, and this is tight.*

Proof. Consider a face $v_0v_1v_2$ in P. We claim that each face must be incident by at least one interior edge. Assume this is not so. Then there is a face $v_0v_1v_2$ of P that has no incident interior edges. It is in some tetrahedron with fourth vertex v_3, and v_0v_3, v_1v_3, v_2v_3 have to be surface edges of P. Therefore the three triangles $v_0v_1v_3$, $v_1v_2v_3$, $v_2v_0v_3$ are either 3-cycles or faces. But 3-cycles are forbidden. If all three triangles are faces, then P is simply a tetrahedron with $n=4$. Therefore our claim holds.

Since there are $2n-4$ faces in a polytope with n vertices, there are at least $2n-4$ interior edges, but each is counted at most 2Δ times since each of the endpoints can be incident to at most Δ faces. Thus $e_m(2\Delta) \geqslant 2n-4$, i.e., $e_m\Delta \geqslant n-2$. This bound can be achieved by considering a bipyramid [11] (Fig. 1(b)), in which $e_m = 1$, $\Delta = n-2$. □

We can generalize Lemma 1 to polytopes having k 3-cycles:

Lemma 2. *For a polytope with k 3-cycles and $n > 4$ vertices, $e_m\Delta \geqslant n-2-3k$.*

Proof. As in Lemma 1, for each of the $2n-4$ faces, there should be at least one incident interior edge, unless, among the three bounding edges of the face, at least one is on a 3-cycle. Each 3-cycle can share an edge with at most six faces (on both sides of the three edges). Thus there remain at least $2n-4-6k$ faces having incident interior edges. With the same argument as in Lemma 1, $e_m(2\Delta) \geqslant 2n-4-6k$, and the result follows. □

The following lemma, which relates the size of triangulations produced by **CutPull** and the number of 3-cycles of a polytope, can easily be deduced from Lemmas 6 and 7 of [5].

Lemma 3. *The* **CutPull** *algorithm produces a triangulation of size at most* $\min(2n-4-\Delta, 2n-7-k)$.

3. Analysis for a special class of polytopes

From the results in [2] and [5], it can be seen that the major problems in finding minimum triangulations appear in 3-cycles and VECSs. In this section we first analyze the special case in which the polytopes concerned have no 3-cycles and no degree-4 vertices (thus no VECSs). Note that the non-existence of 3-cycles implies that there are no degree-3 vertices, and thus all vertices have degrees at least five. We show that in this case the approximation ratio of the **CutPull** algorithm is at most $3/2$, better than the general case ratio $2-\Omega(1/\sqrt{n})$ [5]. Moreover this is tight: we construct polytopes having an approximation ratio no better than $3/2-\varepsilon$ using **CutPull** for any $\varepsilon > 0$. We then consider the case when 3-cycles are present.

Empirically, it has been observed that 3-cycles are not very common in polytopes, in particular those not induced by degree-3 vertices (every degree-3 vertex induces a 3-cycle); and there are certain classes of polytopes, such as prisms, antiprisms, etc. [6] that have no 3-cycles and degrees at least five (provided that the coplanar points are perturbed so that the faces are suitably triangulated, and if necessary with simple modification/replication). Moreover our results also have the following significance:

(i) as far as we know this is one of the very few classes of polytopes that is known to have approximation ratio $2-\varepsilon$ for constant $\varepsilon > 0$. For example, ‘stacked polytopes’ [8] can be triangulated optimally in linear time, or the ‘k-opt polytopes’ [10].
(ii) the existence of 3-cycles and degree-4 vertices can be checked in linear time ([9] and [4] gave linear-time algorithms for enumerating 3-cycles in planar graphs). This is in contrast to k-opt polytopes where no algorithm is known to check whether a polytope is k-opt.
(iii) they may arise as intermediate polytopes in the processing of other triangulation algorithms, e.g., peeling [7].

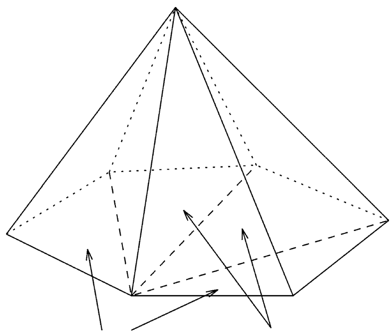

Fig. 2. Ear and internal triangles.

We classify all vertices of a polytope P into two types (w.r.t. a particular triangulation): a vertex is called *type-I* if some interior edge is directly incident to it. Otherwise it is called *type-II*. For any vertex v, we define the *neighborhood* $N(v)$ of v to be the set of vertices directly connected to v on the surface graph, i.e., $N(v) = \{u \mid (u, v) \in \text{surface edges of } P\}$. $N(v)$ forms a 3-dimensional polygon. Consider any triangulation of the polygon $N(v)$. (Note that this is slightly different from the definition of 'dome' or 'cap' [3,5] in that a triangulation of $N(v)$ may not yield a convex patch of triangular faces.) Triangles with two edges on the polygon $N(v)$ are called *'ear' triangles*, and all others are called *'internal' triangles* (Fig. 2).

We present some observations about type-II vertices in the lemma below, which we shall skip the easy proof:

Lemma 4. *Suppose v is a type-*II *vertex of degree d in a polytope P with respect to a triangulation.*

(i) *All tetrahedra incident to v form a triangulation of the region bounded by the* 3*-D polygon $N(v)$ and the faces of P around v. There are $d - 2$ tetrahedra in this part of the triangulation. Their bases triangulate the polygon $N(v)$.*

(ii) *For any triangulation of $N(v)$, if $d \geqslant 5$ and v is not lying on any* 3*-cycles, there is at least one type-*I *vertex in $N(v)$ having two or more incident interior edges. The triangulation of $N(v)$ consists of at least two 'ear' triangles and at least one 'internal' triangle.*

Lemma 5. *For a polytope P having no* 3*-cycles and all vertices have degrees at least five, there are at least $4n/3 - 8/3$ tetrahedra in any triangulation of P.*

Proof. Suppose there are n_1 type-I vertices and n_2 type-II vertices in P, $n_1 + n_2 = n$. We give two different bounds for the size of triangulation:

Bound 1: We want to count the number of interior edge endpoints incident to the vertices (each interior edge having two endpoints). By definition, for each type-I vertex there is at least one interior edge endpoint incident to it. This gives n_1 edge endpoints. In addition, for each of the n_2 type-II vertices, there is at least one type-I vertex in the neighborhood that has two or more edge endpoints incident to it (Lemma 4). But the previous step did not count the extra endpoints (only one endpoint was counted for each type-I vertex). Thus there are at least n_2 additional edge endpoints, if all of them are distinct. It can be shown that at most two type-II vertices share such an additional endpoint; the worst case is as in

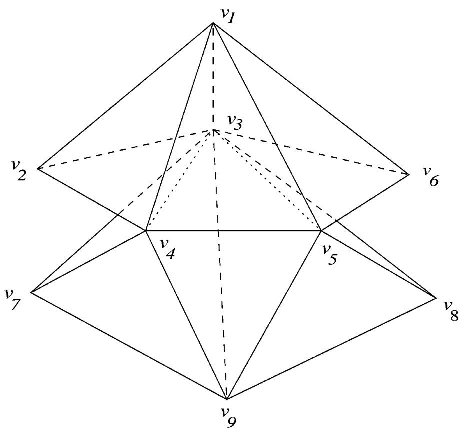

Fig. 3. Two type-II vertices sharing an additional edge.

Fig. 3 where two type-II vertices (v_1 and v_9) sharing a type-I vertex (v_3) that only has two interior edge endpoints. Thus at least $n_2/2$ edge endpoints are added. Since each interior edge has two endpoints to be counted,

$$e_m \geqslant \frac{1}{2}\left(n_1 + \frac{n_2}{2}\right) = \frac{1}{2}\left(n - \frac{n_2}{2}\right).$$

Thus the size of minimum triangulation of P,

$$t_m = e_m + n - 3 \geqslant \frac{3n}{2} - \frac{n_2}{4} - 3.$$

Bound 2: For each type-II vertex v, all tetrahedra incident to it constitute a triangulation of $N(v)$ (Lemma 4) (Fig. 2). Consider any triangulation of the 3-D polygon $N(v)$, with each triangle corresponding to a tetrahedron having v as a vertex. We count the number of tetrahedra incident to the type-II $N(v)$'s. All 'internal' tetrahedra of a type-II vertex v will not be counted by other type-II vertices (since the other three vertices of the tetrahedron are type-I). The 'ear' tetrahedra may be counted twice. For example in Fig. 3 tetrahedra $v_1v_2v_3v_4$ and $v_1v_3v_5v_6$ are 'ear' tetrahedra of a type-II vertex v_1, but the tetrahedra $v_1v_2v_3v_4$ (resp. $v_1v_3v_5v_6$) may also be counted by v_2 (resp. v_6) if v_2 (resp. v_6) are type-II. It cannot be counted more than twice since the other two vertices of the tetrahedron are type-I. Since $d \geqslant 5$, there is at least one 'internal' tetrahedron and at least two 'ear' tetrahedra, giving a total of at least two tetrahedra (each 'ear' counted as 0.5 for this vertex to avoid double counting) incident to each type-II vertex. Thus the total number of tetrahedra incident to these type-II vertices is at least $2n_2$.

Considering both bounds, the number of tetrahedra is at least $\max(3n/2 - n_2/4 - 3, 2n_2)$. Since the two expressions are decreasing and increasing functions of n_2, respectively, the maximum is minimized when the expressions are equal, i.e., $n_2 = (2n - 4)/3$, and $t_m \geqslant 4n/3 - 8/3$. □

The above bound is tight (up to a constant additive factor) as shown below:

Lemma 6. *There exist polytopes without* 3*-cycles and degree*-4 *vertices, and with constant vertex degrees, such that the sizes of minimum triangulations are at most* $4n/3 + 8/3$.

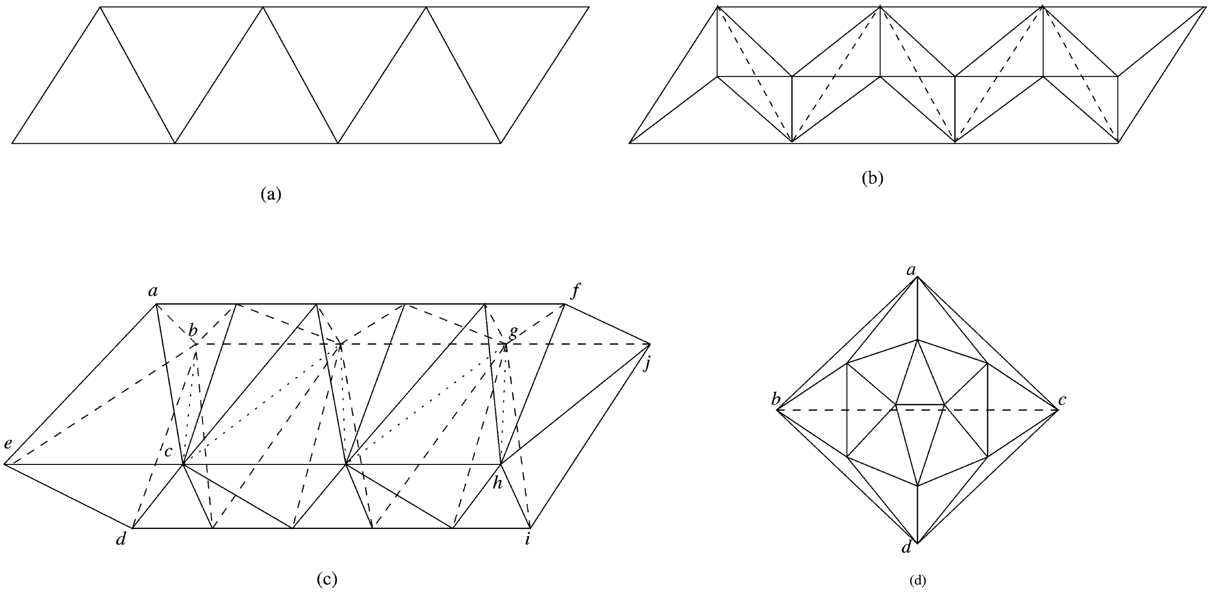

Fig. 4. The construction. (a) m triangles in the horizontal plane. (b) Tetrahedra placed above the m triangles (top view). (c) The resulting polytope (3D view), with two ends to be further processed. (d) Patch structure. The two triangles at the back (as indicated by the dashed line) are to be attached to newly exposed faces.

Proof. Consider a sequence of m triangles in a horizontal plane (Fig. 4(a)). We place a tetrahedron on top of and below every triangle. Then for every two adjacent tetrahedra on the top half, connect their top vertices, thus ‘filling the valley’ between the two tetrahedra with another tetrahedron. The bottom half is treated similarly. This gives a convex polytope with $3m+2$ vertices and triangulated by $4m-2$ tetrahedra (Fig. 4(b,c)). All the vertices in the polytope have degrees at least 5, except the vertices at the two ends, labelled a, d, e, f, i, j in Fig. 4(c). We handle them as follows. For the left end, we remove the tetrahedra $abce$ and $bcde$, leaving a non-convex polytope with bc being a non-convex edge, and abc, bcd being two newly exposed faces. We patch the structure in Fig. 4(d) to cover those newly exposed faces, while maintaining convexity of the polytope. This structure has 12 vertices (8 of them are new vertices when patched), all with degrees at least five. This patched part itself is a convex polytope triangulable using at most $2n-4-5 = 2(12)-9 = 15$ additional tetrahedra (by pulling). Apply the same to the right end. The resulting polytope has $3m+2+2(-1+8) = 3m+16$ vertices, no 3-cycles, and can be triangulated using at most $4m-2+2(-2+15) = 4m+24$ tetrahedra. The bound follows. □

The tight bound on the size of triangulations gives a tight bound on the approximation ratio of **CutPull**:

Theorem 1. *The approximation ratio of* **CutPull** *algorithm for polytope without* 3*-cycles and all vertices having degrees at least five is at most* 3/2*, and this is tight.*

Proof. The bound on the approximation ratio follows from Lemmas 3 and 5:

$$r \leqslant \frac{2n-7}{4n/3-8/3} < \frac{3}{2}.$$

That the bound is tight follows from the constructed polytope in Lemma 6, having constant vertex degrees, no 3-cycles, and $t_m \leqslant 4n/3 + 8/3$. Thus for those polytopes, **CutPull** gives

$$r \geqslant \frac{2n - 4 - \Delta}{4n/3 + 8/3} = \frac{3}{2} - \varepsilon$$

where $\varepsilon = \Theta(1/n)$ tends to 0 as n tends to infinity. □

With the presence of 3-cycles (but still without degree-3 and degree-4 vertices), we have:

Theorem 2. *For polytopes with k 3-cycles and all vertices have degrees at least five,* **CutPull** *gives an approximation ratio*

$$r \leqslant \frac{2n - 7 - k}{\max(n - 3, (4n - 8)/3 - 4k)} < 2 - \frac{1}{12}$$

for any k.

Proof. The argument in Lemma 5 works for vertices not lying on any 3-cycles. Suppose there are n' vertices not lying on 3-cycles. We have $n' \geqslant n - 3k$, so $t_m \geqslant 4(n - 3k)/3 - 8/3 = (4n - 8)/3 - 4k$. With Lemma 3 we have the approximation ratio

$$r \leqslant \frac{2n - 7 - k}{\max(n - 3, (4n - 8)/3 - 4k)}.$$

Note that $n - 3 \geqslant (4n - 8)/3 - 4k$ if and only if $k \geqslant (n + 1)/12$. So if $k \geqslant (n + 1)/12$, we have

$$r \leqslant \frac{2n - 7 - k}{n - 3} \leqslant \frac{2n - 7 - (n + 1)/12}{n - 3} < 2 - \frac{1}{12}.$$

If $k < (n + 1)/12$, we have

$$r \leqslant \frac{2n - 7 - k}{(4n - 8)/3 - 4k}$$

and since the value of this fraction increases with k, we have

$$r \leqslant \frac{2n - 7 - (n + 1)/12}{(4n - 8)/3 - 4(n + 1)/12} < 2 - \frac{1}{12}.$$

Thus the ratio is at most $2 - 1/12$ for any k. □

4. Analysis for polytopes with bounded vertex degrees

In this section, we consider convex polytopes with vertex degrees bounded above by a given Δ. We show that in this case the **CutPull** algorithm can be applied with improved approximation ratio, and the ratio is tight up to combinatorial considerations. The analysis generalizes that in [5] by incorporating Δ in the bound. This is useful when Δ is small or has known asymptotic behaviour. In particular, we can improve the approximation ratio when the vertex degrees are bounded above by a constant. This occurs frequently, for example, in randomly generated polytopes.

4.1. Upper bound

Theorem 3. *The* **CutPull** *algorithm gives an approximation ratio of* $2 - \Omega(1/\Delta) - \Omega(\Delta/n)$.

Proof. Without loss of generality assume $\Delta \geqslant 4$. Recall that k is the number of 3-cycles. In the following we will repeatedly make use of the following inequality: if $A, B > 0$ and $A/B < 2$ then $\frac{A}{B} < \frac{A+2x}{B+x}$ for $x > 0$. We consider two cases.

Case 1: $k + 1 \leqslant \Delta/6$. Then from Lemmas 2 and 3

$$\begin{aligned} r &\leqslant \frac{2n-4-\Delta}{e_m+n-3} \leqslant \frac{2n-4-\Delta}{(n-2-3k)/\Delta+n-3} = \frac{2n\Delta-\Delta(\Delta+4)}{(1+\Delta)n-2-3k-3\Delta} \\ &< \frac{2n\Delta-\Delta(\Delta+4)+6\Delta+6k+4}{(1+\Delta)n} = \frac{2\Delta}{1+\Delta} - \frac{\Delta(\Delta+1)-3\Delta}{(1+\Delta)n} + \frac{6k+4}{(1+\Delta)n} \\ &< \frac{2\Delta}{1+\Delta} - \frac{\Delta}{n} + \frac{3}{n} + \frac{6(k+1)}{(1+\Delta)n} \leqslant \frac{2\Delta}{1+\Delta} - \frac{\Delta-3}{n} + \frac{\Delta}{5n} = 2 - \frac{2}{\Delta+1} - \frac{0.8\Delta-3}{n}. \end{aligned}$$

Case 2: $k + 1 > \Delta/6$. Then from Lemmas 2 and 3

$$\begin{aligned} r &\leqslant \frac{2n-7-k}{e_m+n-3} < \frac{2n-(k+1)}{(n-2-3k)/\Delta+n} = \frac{2n\Delta-\Delta(k+1)}{(1+\Delta)n-(3k+2)} < \frac{2n\Delta-\Delta(k+1)+(6k+4)}{(1+\Delta)n} \\ &= \frac{2\Delta}{\Delta+1} - \frac{(\Delta-6)k+(\Delta-4)}{(\Delta+1)n} < \frac{2\Delta}{\Delta+1} - \frac{(\Delta-6)(\Delta/6-1)+(\Delta-4)}{(\Delta+1)n} \\ &= 2 - \frac{2}{\Delta+1} - \Omega\left(\frac{\Delta}{n}\right). \qquad \square \end{aligned}$$

It can be seen that the worst case occurs when $\Delta = \Theta(\sqrt{n})$ in which the bound reduces to $2 - \Omega(1/\sqrt{n})$ in [5].

When the maximum degree Δ is bounded by a constant, Theorem 3 shows an improved approximation ratio:

Corollary 1. *When* Δ *is constant, the* **CutPull** *algorithm gives an approximation ratio no larger than* $2 - \frac{2}{\Delta+1}$. *For example, the ratio is* $12/7$ *for* $\Delta = 6$ *and* $7/4$ *for* $\Delta = 7$.

4.2. Lower bound

It is proved in [5] that no algorithm that only considers the combinatorial structures of polytopes can give an approximation ratio better than $2 - O(1/\sqrt{n})$ for the minimum triangulation problem. The proof is based on constructing two polytopes P1 and P2 with the same combinatorial structure but having different sizes in their minimum triangulations. In this subsection we prove a more general result when the maximum degree Δ of the polytope is given. This shows that our upper bound in the previous subsection is asymptotically tight when only combinatorial information is considered.

We construct two combinatorially equivalent polytopes P1 and P2. The construction is similar to what is shown in [5] except that in [5] some vertices have unbounded degrees. In our construction, we have to bound the maximum degree of all vertices.

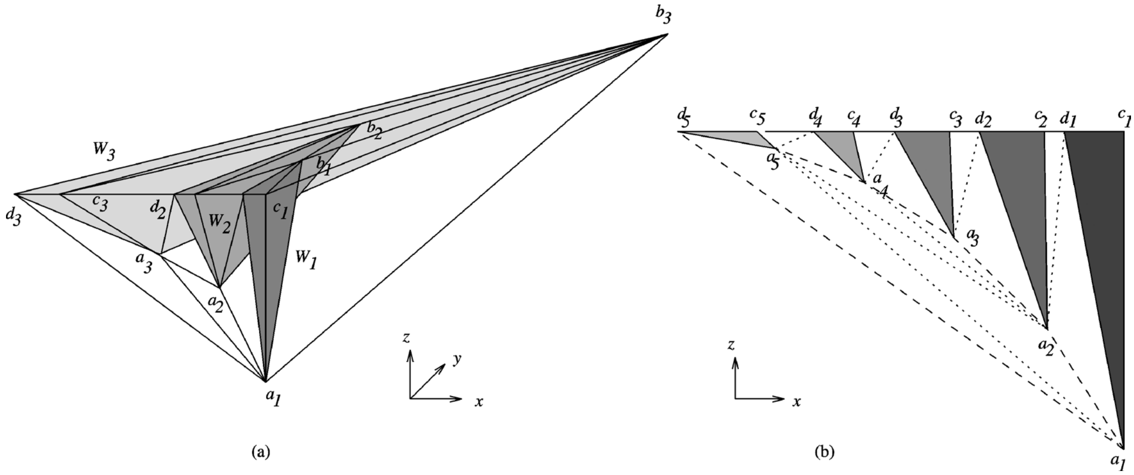

Fig. 5. (a) Construction of P1 and P2, showing 3 wedges. q_i's are not shown. (b) The wedges with $m = 5$, showing the zig-zag paths on the vertical plane.

First, a set of m VECSs each of size s are placed as in Fig. 5(a). Wedge W_i has vertices (a_i, b_i, c_i, d_i) with $a_i b_i$ being the main diagonal. All main diagonals pass through the origin. All faces $a_i c_i d_i$ lie on the vertical plane $y = -1$ while all faces $b_i c_i d_i$ lie on the horizontal plane $z = 1$. Vertices $q_i^1, \ldots, q_i^s$ are added between c_i and d_i for each wedge to form a VECS. The a_i's form a convex chain w.r.t. (0, -1, $-\infty$), and the b_i's form a convex chain w.r.t. (∞, 0, 1). We have $n = m(s + 4)$.

Second, notice that all vertices lie on two planes, violating the general position assumption. We remove this degeneracy by perturbing the vertices slightly, so that the polytope has the following set of edges (see Fig. 5):

$$a_k c_k, \quad a_k d_k, \quad b_k c_k, \quad b_k d_k \quad (1 \leqslant k \leqslant m);$$
$$d_k a_{k+1}, \quad d_k c_{k+1}, \quad b_k c_{k+1}, \quad a_k a_{k+1}, \quad b_k b_{k+1} \quad (1 \leqslant k \leqslant m-1);$$
$$q_k^i q_k^{i+1}, \quad q_k^1 c_k, \quad q_k^s d_k \quad (1 \leqslant i \leqslant s-1,\ 1 \leqslant k \leqslant m);$$
$$q_k^i a_k, \quad q_k^i b_k \quad (1 \leqslant i \leqslant s,\ 1 \leqslant k \leqslant m);$$
$$c_1 b_m, \quad a_1 b_m, \quad a_1 d_m.$$

To cope with our constant-degree construction, the a_i's and b_i's are connected together in a zig-zag manner, i.e., $a_1 a_n a_2 a_{n-1}, \ldots, b_1 b_n b_2 b_{n-1}, \ldots$ (Fig. 5(b)). It is easy to show that in this construction, the maximum degree $\Delta = s + 7$ (attained at, e.g., a_2 in Fig. 5(b)), and we can apply sufficiently small perturbations to the vertices so that they are in general position.

Now the main diagonals all intersect at the origin. In the third step for P1, we 'push' the wedges towards each other slightly so that all wedges intersect each other. For P2, we shrink the wedges slightly so that they do not intersect. The exact details can be found in [5]. In this way, P1 will have a large size of triangulation because the wedges are 'interlocked' (i.e., penetrating each other), while P2 can have a small triangulation, although the two have the same combinatorial structure.

Theorem 4. *Any triangulation algorithm that only considers the combinatorial structure of a convex polytope cannot give an approximation ratio better than* $2 - O(1/\Delta) - O(\Delta/n)$.

Proof. We first show that any minimum triangulation of P1 has at least $(\frac{2\Delta-14}{\Delta-3})n-\Delta$ tetrahedra, while any minimum triangulation of P2 has at most $(\frac{\Delta+1}{\Delta-3})n$ tetrahedra.

As discussed in Section 1, the wedges have the property that they admit triangulation of size either at most $s+1$ or at least $2s$, depending on the presence of their 'main diagonal' in the triangulation. For P1, at most one main diagonal of these m wedges can be present in any triangulation. Thus

$$t_{\mathrm{P1}} \geqslant (m-1)(2s)+(s+1)=\left(\frac{n}{\Delta-3}-1\right)(2\Delta-14)+(\Delta-6)>\left(\frac{2\Delta-14}{\Delta-3}\right)n-\Delta.$$

For P2, each wedge can be triangulated into $s+1$ tetrahedra using their main diagonals. Removing these wedges leaves a non-convex region. This can be triangulated into $4(m-1)+3(m-2)+2=7m-8$ tetrahedra, using the 'shielding' argument same as that in [5]; due to space limitation we do not repeat it here. Note that the a_i's and b_i's have to be 'zig-zagged' in a matching manner for the proof to work. So

$$t_{\mathrm{P2}} \leqslant m(s+1)+7m-8=\left(\frac{n}{s+4}\right)(s+1+7)-8<\left(\frac{\Delta+1}{\Delta-3}\right)n.$$

An algorithm that only considers combinatorial structures cannot distinguish P1 and P2, and always has to give the triangulation of larger size. With the above bounds, we thus have

$$r \geqslant \frac{(\frac{2\Delta-14}{\Delta-3})n-\Delta}{(\frac{\Delta+1}{\Delta-3})n}=\frac{2\Delta-14}{\Delta+1}-\frac{\Delta(\Delta-3)}{n(\Delta+1)}=2-\mathrm{O}\left(\frac{1}{\Delta}\right)-\mathrm{O}\left(\frac{\Delta}{n}\right). \qquad \square$$

5. Conclusion

We gave improved approximation ratios for the minimum polytope triangulation problem for two special classes of polytopes: one having no 3-cycles and no degree-4 vertices, and one with bounded maximum vertex degrees. For the case without 3-cycles and degree-4 vertices, our algorithm gives a ratio of $3/2$. This seems to be a rather restricted class of polytopes; can it be optimally triangulated in polynomial time? Can we identify the (more restricted?) class of polytopes which our algorithm will give the optimal triangulation? Stacked polytopes are one known type. Can we identify classes of polytopes that can be triangulated optimally or near-optimally in polynomial time, using perhaps other algorithms? The results may also be generalized to polytopes having few (but nonzero) degree-4 vertices.

For the constant degree case, we get an asymptotically tight approximation ratio $2-\Omega(1/\Delta)-\Omega(\Delta/n)$, the lower bound being established if only combinatorial structure is considered. It is actually not known whether the constant-degree case is NP-hard (like the general-degree case), and what happens when non-combinatorial information is considered.

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