Greedy Online Frequency Allocation in Cellular Networks

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Abstract

The online frequency allocation problem for cellular networks has been well studied in these years. Given a mobile telephone network, whose geographical coverage area is divided into cells, phone calls are served by assigning frequencies to them, and no two calls emanating from the same or neighboring cells are assigned the same frequency. Assuming an online setting that the calls arrive one by one, the problem is to minimize the span of the frequencies used.

In this paper, we study the greedy approach for the online frequency allocation problem, which assigns the minimal available frequency to a new call so that the call does not interfere with calls of the same cell or neighboring cells. If the calls have infinite duration, the competitive ratio of greedy algorithm has a tight upper bound of 17/7, which closes the gap of $[17/7, 2.5)$ in [3]. If the calls have finite duration, i.e., each call may be terminated at some time, the competitive ratio of the greedy algorithm has a tight upper bound of 3.

Keywords: Online Frequency Allocation, Competitive Analysis, Greedy Algorithm, Cellular Network

1 Introduction

Wireless Communication based on Frequency Division Multiplexing (FDM) technology is widely used in the area of mobile telephony. In the network of wireless communication, the geographic area is divided into small cellular regions [8] or *cells* as shown in Fig. 1, each containing one base station. Each base station serves the calls in its cell via radio frequencies, and base stations communicate with each other through a high-speed wired network. To avoid radio interference, the same frequency cannot be assigned to two different calls emanating from the same cell or neighboring cells. Since the frequency spectrum is a scarce resource, we should reuse the same frequency for different calls in the cells not close to each other. Efficient utilization of the available spectrum is very important to the *frequency allocation* problem [1, 3, 4, 6, 7, 9, 10].

Frequency Allocation Problem. In this paper, we focus on the online version of the frequency allocation problem, in which a sequence σ of calls arrive over time, where $\sigma = (C_1, C_2, \ldots, C_k, \ldots,)$ and C_k represents the cell from which the k-th call emanates. Each call C_k must be assigned upon its arrival, without information about future calls $\{C_i | i > k\}$, a frequency $\mathcal{A}(C_k) \in Z^+$ where $Z^+ = \{1, 2, \ldots\}$ of available frequencies, that is different from those of other

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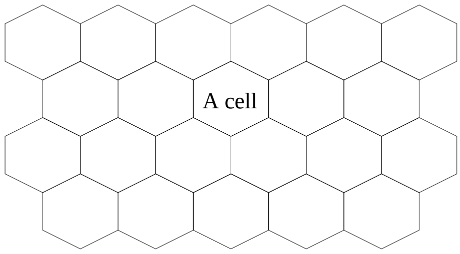


Figure 1: a description of cellular network

calls in the same cell or neighboring cells, i.e., $\mathcal{A}(C_k) \neq \mathcal{A}(C_i)$, where $i < k$ and C_i is adjacent to C_k or the same as C_k. The integer frequency once assigned to a call cannot be changed during the survival of this call. The *online frequency allocation problem for cellular network* (FAC for short) is to minimize the maximum assigned frequency, i.e. $\max\{\mathcal{A}(C_k)|k = 1, 2, \ldots, n\}$. If all the information of C_k is known in advance, we call this problem *off-line frequency allocation problem*, which is NP-hard [9].

Two models of online frequency allocation problems will be investigated. The first model is that all the calls have infinite duration [3]. We call this model *frequency allocation without deletion*. The second model is that each call may be terminated at some time, i.e., each call is characterized by two parameters: arrival time and termination time. However, the termination time is not known when the call arrives online. We call this model *frequency allocation with deletion*.

Performance Measures. We use competitive analysis [2] to measure the performances of online algorithms. To serve all the calls in a given sequence σ, $\mathcal{A}(\sigma)$ denotes the highest frequency used by the online algorithm $\mathcal{A}$, and $\mathcal{O}(\sigma)$ denotes the highest frequency used by the optimal off-line algorithm.

The *competitive ratio* of algorithm $\mathcal{A}$ is defined as

$$R_{\mathcal{A}} = \sup_{\sigma} \frac{\mathcal{A}(\sigma)}{\mathcal{O}(\sigma)}.$$

Known Results. Previous results have mainly focused on the *without-deletion* model. A simple strategy for frequency allocation problem is the fixed allocation assignment (FAA) [8], in which cells are partitioned into independent sets and independent sets are each assigned a separate set of frequencies. FAA gives an easy upper bound of 3 for FAC.

Another intuitive approach is the *greedy algorithm* (Greedy), which assigns the minimal available frequency to a new call so that the call does not interfere with calls of the same cell or neighboring cells. Caragiannis et al [3] proved that the competitive ratio of Greedy for FAC is at least 17/7 and at most 2.5.

In the *with-deletion* model, the general lower bound of competitive ratio is 2[5].

Our Contributions. In this paper, we analyze Greedy in both of the *without-deletion* and *with-deletion* models. In the *without-deletion* model, we tighten the upper bound of Greedy to 17/7. In the *with-deletion* model, we prove that both the upper bound and the lower bound of Greedy are 3. Thus, 3 is the best possible competitive ratio for Greedy in *with-deletion* model.

2 Greedy in the Without-Deletion Model

In this section we give a tighter analysis of Greedy and show that Greedy is 17/7-competitive for cellular networks, which matches the lower bound of Greedy as given in [3].

Theorem 1. *Greedy for FAC has a competitive ratio of* $17/7$ *in the without-deletion model.*

Proof. Suppose the highest frequency h used by Greedy is assigned to a call from a cell n_0 after which no more calls are made. Let the six neighboring cells of n_0 be n_i for $1 \leq i \leq 6$ in clockwise order as shown in Figure 2. Let f^* be the maximum number of calls from any three adjacent cells. Let $C(N)$ denote the total number of calls from N where N can be an individual cell or a set of cells. Without loss of generality, we can assume that the three adjacent cells n_0, n_1, and n_2 together have $C(\{n_0, n_1, n_2\}) = f^*$ calls; otherwise, the adversary may initiate more calls from n_0 to get a larger h and thereby increase the competitive ratio.

Let $C(n_1) = i$, $C(n_2) = i + j$, and $C(n_0) = i + k$, for some $i, j \geq 0$ and k may be positive or negative. Hence, we have $f^* = 3i + j + k$. Consider another three adjacent cells, say n_0, n_2 and n_3, $C(\{n_0, n_2, n_3\}) \leq f^*$. Thus, we can assume $C(n_3) = i - c_1$ for some $c_1 \geq 0$. Similarly, we can deduce the numbers of calls from other cells as follows.

$$\begin{array}{llll} C(n_0) = i + k, & C(n_1) = i, & C(n_2) = i + j, & C(n_3) = i - c_1 \\ C(n_4) = i + j - c_2, & C(n_5) = i - c_3, & C(n_6) = i + j - c_4 & \\ \multicolumn{4}{l}{\quad \text{for some } i, j, c_1, c_4 \geq 0, \text{ and } c_1 + c_2, c_2 + c_3, c_3 + c_4 \geq 0} \end{array} \tag{1}$$

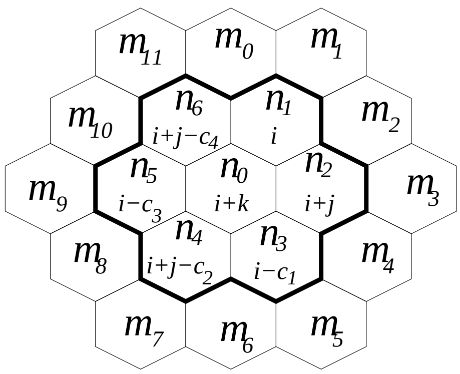


Figure 2: Cellular network for upper bound of Greedy

Let d be the number of distinct frequencies among the frequencies assigned to the calls from n_0 and all n_0's neighbors. Let $r = \sum_{i=0}^{6} C(n_i) - d = 7i + 3j + k - \sum_{i=1}^{4} c_i - d$, i.e., r is the total number of calls with "reused" frequencies in these seven cells. Recall that h is the highest frequency used by Greedy and f^* is the maximum number of calls from any three adjacent cells. The competitive ratio of Greedy is at most h/f^*. In the following we prove that $h/f^* \leq 17/7$. Since $h = d = \sum_{i=0}^{6} C(n_i) - r = 7i + 3j + k - \sum_{i=1}^{4} c_i - r$ and $f^* = 3i + j + k$, it suffices to prove that

$$\frac{7i + 3j + k - \sum_{i=1}^{4} c_i - r}{3i + j + k} \leq \frac{17}{7},$$

or after rewriting the inequality,

$$2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \geq 4j.$$

This inequality is proved in the following lemma. □

Lemma 2. $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \geq 4j$.

Proof. Let M denote the set of cells, excluding n_0, that are neighbors of one of the cells n_i for $1 \leq i \leq 6$ but not the cells n_i themselves. Assume the cells in M are labeled as m_i for $0 \leq i \leq 11$ in clockwise order, starting with m_0 which is the neighbor of both n_1 and n_6 (Figure 2).

Let s_i denote the total number of calls from cells in M which are the neighbors of n_i. Precisely, $s_i = C(\{m_{2i-2}, m_{2i-1}, m_{2i}\})$. Note that the indices are in modulo 12, e.g., $s_6 = C(\{m_{10}, m_{11}, m_0\})$. We can deduce the value s_i for $1 \leq i \leq 6$ as follows. For example, where s_4 is considered, we have $C(m_6) \leq f^* - C(\{n_3, n_4\}) = i + k + c_1 + c_2$ because m_6, n_3 and n_4 are three adjacent cells and similarly $C(\{m_7, m_8\}) \leq f^* - C(n_4) = 2i + k + c_2$. Thus $s_4 = C(\{m_6, m_7, m_8\}) \leq 3i + 2k + 2c_2 + c_1$. On the other hand, we have $C(m_8) \leq f^* - C(\{n_4, n_5\}) = i + k + c_2 + c_3$ and $C(\{m_6, m_7\}) \leq f^* - C(n_4) = 2i + k + c_2$, and hence $s_4 \leq 3i + 2k + 2c_2 + c_3$. Since s_4 has to satisfy both inequalities, we have $s_4 \leq 3i + 2k + 2c_2 + \min(c_1, c_3)$. Similarly, we can deduce the inequalities for other s_i as follows.

$$\begin{aligned}
&s_1 \leq 3i + j + 2k, && s_2 \leq 3i + 2k, \\
&s_3 \leq 3i + j + 2k + 2c_1, && s_4 \leq 3i + 2k + 2c_2 + \min(c_1, c_3), \\
&s_5 \leq 3i + j + 2k + 2c_3 + \min(c_2, c_4), && s_6 \leq 3i + 2k + 2c_4 + \min(0, c_3).
\end{aligned} \tag{2}$$

Let $H(N)$ denote the highest frequency assigned to calls from N where N can be an individual cell or a set of cells. In the following, we study the six different cases where $H(\{n_1, n_2, \ldots, n_6\}) = H(n_i)$ for $1 \leq i \leq 6$. In each case, we show that $2i+10k+7\sum_{i=1}^{4} c_i+7r \geq 4j$.

Case 1: $H(\{n_1, n_2, \ldots, n_6\}) = H(n_1)$: By the greedy algorithm, when frequency $H(n_1)$ is assigned to a call from n_1, all the frequencies less than $H(n_1)$ must have been assigned to calls from n_1 or neighbors of n_1. Those frequencies in n_3, n_4, and n_5 but not duplicated in n_1, n_2 and n_6 must be assigned to calls from m_0, m_1 or m_2, which are the other neighbors of n_1. Hence, we have (from Eq. (1) and (2))

$$\begin{aligned}
& C(\{n_3, n_4, n_5\}) - r \leq s_1 = 3i + j + 2k \\
\Rightarrow \quad & 3i + j - c_1 - c_2 - c_3 - r \leq 3i + j + 2k \\
\Rightarrow \quad & 2k + c_1 + c_2 + c_3 + r \geq 0.
\end{aligned} \tag{3}$$

We further consider three sub-cases: $H(\{n_2, n_4, n_6\}) = H(n_2)$, $H(\{n_2, n_4, n_6\}) = H(n_4)$, and $H(\{n_2, n_4, n_6\}) = H(n_6)$.

For $H(\{n_2, n_4, n_6\}) = H(n_2)$**,** the frequencies in n_4 and n_6 but not duplicated in n_1, n_2 and n_3 must be assigned to calls from m_2, m_3 or m_4. Thus we have $C(\{n_4, n_6\}) - r \leq s_2$, which implies that $2i + 2j - c_2 - c_4 - r \leq 3i + 2k$ (from Eq. (1) and (2)), i.e.,

$$i + 2k + c_2 + c_4 + r \geq 2j. \tag{4}$$

As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \geq 2(i + 2k + c_2 + c_4 + r) + 3(2k + c_1 + c_2 + c_3 + r) + 4c_1 + 2(c_2 + c_3) + 2(c_3 + c_4) + 3c_4 + 2r \geq 4j$ because the first term is at least $4j$ (from Eq. (4)) and all the other terms are non-negative (from Eq. (1) and (3)).

For $H(\{n_2, n_4, n_6\}) = H(n_4)$**,** the frequencies in n_2 and n_6 but not duplicated in n_3, n_4 and n_5 must be assigned to calls from m_6, m_7 or m_8. Thus we have $C(\{n_2, n_6\}) - r \leq s_4$, which implies that $2i + 2j - c_4 - r \leq 3i + 2k + 2c_2 + \min(c_1, c_3)$ (from Eq. (1) and (2)), i.e.,

$$i + 2k + 2c_2 + c_4 + \min(c_1, c_3) + r \geq 2j. \quad (5)$$

As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \geq 2(i + 2k + 2c_2 + c_4 + \min(c_1, c_3) + r) + 3(2k + c_1 + c_2 + c_3 + r) + 4c_1 + 2(c_3 - \min(c_1, c_3)) + 2(c_3 + c_4) + 3c_4 + 2r \geq 4j$ because the first term is at least $4j$ (from Eq. (5)) and all the other terms are non-negative (from Eq. (1) and (3)).

For $H(\{n_2, n_4, n_6\}) = H(n_6)$**,** the frequencies in n_2 and n_4 but not duplicated in n_1, n_5 and n_6 must be assigned to calls from m_{10}, m_{11} or m_0. Thus we have $C(\{n_2, n_4\}) - r \leq s_6$, which implies that $2i + 2j - c_2 - r \leq 3i + 2k + 2c_4 + \min(0, c_3)$ (from Eq. (1) and (2)), i.e.,

$$i + 2k + c_2 + 2c_4 + \min(0, c_3) + r \geq 2j. \quad (6)$$

As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \geq 2(i + 2k + c_2 + 2c_4 + \min(0, c_3) + r) + 3(2k + c_1 + c_2 + c_3 + r) + 4c_1 + 2(c_2 + c_3) + 2(c_3 - \min(0, c_3)) + 3c_4 \geq 4j$ because the first term is at least $4j$ (from Eq. (6)) and all the other terms are non-negative (from Eq. (1) and (3)).

Thus, the lemma is proved for this case when $H(\{n_1, n_2, \ldots, n_6\}) = H(n_1)$.

Case 2: $\boldsymbol{H(\{n_1, n_2, \ldots, n_6\}) = H(n_2)}$**.** Similar to the previous case, we can deduce that $C(\{n_4, n_5, n_6\}) - r \leq s_2$, i.e.,

$$2k + c_2 + c_3 + c_4 + r \geq 2j.$$

Hence, $2i+10k+7\sum_{i=1}^{4} c_i+7r \geq 2i+5(2k+c_2+c_3+c_4+r)+5c_1+2(c_1+c_2)+2(c_3+c_4)+2r \geq 4j$ because the second term is at least $10j$ and all the other terms are non-negative.

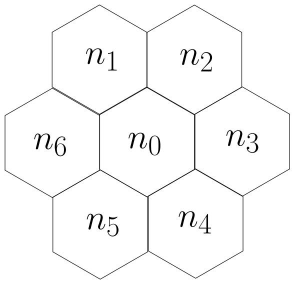
For each of the remaining cases: we can also prove that $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \geq 4j$. The details are given in the appendix. □

3 Greedy in the With-Deletion Model

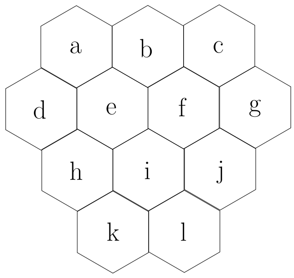
In this section, we analyze Greedy in the *with-deletion* model, i.e., each call may be terminated at some time. We prove that the upper bound and lower bound of Greedy are both 3. Thus, the best possible competitive ratio is 3 for Greedy in the *with-deletion* model.

Theorem 3. *Greedy for FAC is* 3*-competitive in the with-deletion model.*

Proof. Consider Fig. 3(a), assume the highest frequency f appears in cell n_0. Let x be the number of frequencies in cell n_0, y be the number of distinct frequencies in cells n_1 to n_6. Since Greedy is to choose the smallest frequency without interference with the neighboring frequencies, those frequencies less than f must all appear in n_0 or n_i,$(1 \leq i \leq 6)$. Therefore, we can say that



(a) Upper bound



(b) Lower bound

Figure 3: Upper bound and Lower bound of Greedy in cellular network

$f = x + y$. In this configuration, optimal allocation uses at least $x + y/3$ frequencies. Hence, the competitive ratio is at most

$$\frac{x+y}{x+y/3} < 3.$$

The following is to prove by induction that 3-competitive is best possible for Greedy.

Consider Fig. 3(b), define *tri-group* to be the set of three cells which are "one edge" away from each other, e.g., cell sets $\{a, f, h\}$ and $\{b, g, i\}$.

We prove the following hypothesis by induction. When the highest frequency $3n + 1$ is assigned to a call in cell x, all frequencies from 1 to $3n$ would have been used in a *tri-group*, e.g., $\{u, v, w\}$, around x, s.t. $C(u) = C(v) = C(w) = n$. As for the optimal off-line algorithm, $n + 1$ frequencies could be sufficient to take up all these calls.

Base Step $n = 1$. When one call appears at cells c, d and i, Greedy uses the frequency 1 to these calls. Then the frequency 2 will be assigned to a call at cells a and cell g, and the frequency 3 to a call at cell b. Delete the call in cell c. At last, the frequency 4 has to be assigned to a call at cell f, since the frequencies 1, 2 and 3 are all around f.

In this step, Greedy uses 4 frequencies while the optimal off-line assignment uses only two frequencies, and each cell of *tri-group* $\{b, g, i\}$ contains one frequency.

Induction Step Suppose the hypothesis is true for $n = k$. Now we prove that it is also correct for $n = k + 1$.

By the induction hypothesis, Greedy can force the frequency $3k + 1$ to a call C_1 in cell d and another call C_2 in cell g. Terminate all other calls except C_1 and C_2. From the hypothesis, Greedy can assign the frequencies 1 to $3k$ to appear in *tri-group* $\{a, f, h\}$, and each cell contains k frequencies. Then a call C_3 arrives at e, Greedy must use the frequency $3k + 2$. Then terminate all other calls except C_3 in e and C_2 in g. After that, Greedy forces the frequencies 1 to $3k$ to appear in *tri-group* $\{b, g, i\}$, s.t. $C(b) = C(i) = k$ and $C(g) = k + 1$(including the call C_2). Now a call C_4 arrives at f, Greedy must use the frequency $3k + 3$ to this call. Then terminate all other calls except C_4 in f. By similar description, Greedy can assign the frequency $3k + 1$ to a call C_5 in h and the frequency $3k + 2$ to a call C_6 in l. Then force the frequencies 1 to $3k$ to appear in *tri-group* $\{f, h, l\}$, s.t. $C(f) = C(h) = C(l) = 3k + 1$. Finally, one call C_7 arrives at i, Greedy must use the frequency $3k + 4$.

With the induction hypothesis on the above sequence, the optimal off-line algorithm can assign frequencies $k+1$, $k+2$, $k+2$, $k+1$, $k+1$, $k+1$ and $k+2$ to the calls C_1 to C_7 respectively.

Thus, to satisfy the call sequence described above, Greedy must use $3k+4$ frequencies, while the optimal off-line assignment uses only $k+2$ frequencies. Therefore, the competitive ratio of Greedy in cellular network is at least

$$\frac{3n-2}{n} \longrightarrow 3.$$

□

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Appendix

Remaining proof of Lemma 2

We continue for the remaining cases, i.e., $H(\{n_1, n_2, \ldots, n_6\}) = H(n_i)$ for $3 \le i \le 6$.

Case 3: $\boldsymbol{H(\{n_1, n_2, \ldots, n_6\}) = H(n_3)}$. Similar to Case 1, we have

$$\begin{aligned} & C(\{n_1, n_5, n_6\}) - r \le s_3 = 3i + j + 2k + 2c_1 \\ \Rightarrow\ & 3i + j - c_3 - c_4 - r \le 3i + j + 2k + 2c_1 \\ \Rightarrow\ & 2k + 2c_1 + c_3 + c_4 + r \ge 0. \end{aligned}$$

We further consider three sub-cases: $H(\{n_2, n_4, n_6\}) = H(n_2)$, $H(\{n_2, n_4, n_6\}) = H(n_4)$, and $H(\{n_2, n_4, n_6\}) = H(n_6)$.

For $H(\{n_2, n_4, n_6\}) = H(n_2)$**,** as in case 1, we have $i + 2k + c_2 + c_4 + r \ge 2j$. As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \ge 2(i + 2k + c_2 + c_4 + r) + 3(2k + 2c_1 + c_3 + c_4 + r) + (c_1 + c_2) + 4(c_2 + c_3) + 2c_4 + 2r \ge 4j$ because the first term is at least $4j$ and all the other terms are non-negative.

For $H(\{n_2, n_4, n_6\}) = H(n_4)$**,** as in case 1, we have $i + 2k + 2c_2 + c_4 + \min(c_1, c_3) + r \ge 2j$. As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \ge 2(i + 2k + 2c_2 + c_4 + \min(c_1, c_3) + r) + 3(2k + 2c_1 + c_3 + c_4 + r) + (c_1 + c_3 - 2\min(c_1, c_3)) + 3(c_2 + c_3) + 2c_4 + 2r \ge 4j$ because the first term is at least $4j$ and all the other terms are non-negative.

For $H(\{n_2, n_4, n_6\}) = H(n_6)$**,** as in case 1, we have $i + 2k + c_2 + 2c_4 + \min(0, c_3) + r \ge 2j$. As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \ge 2(i + 2k + c_2 + 2c_4 + \min(0, c_3) + r) + 3(2k + 2c_1 + c_3 + c_4 + r) + (c_1 + c_2) + 4(c_2 + c_3) - 2\min(0, c_3) + 2r \ge 4j$ because the first term is at least $4j$ and all the other terms are non-negative.

Case 4: $\boldsymbol{H(\{n_1, n_2, \ldots, n_6\}) = H(n_4)}$. Similar to the previous cases, we can deduce that $C(\{n_1, n_2, n_6\}) - r \le s_4 = 3i + 2k + 2c_2 + \min\{c_1, c_3\}$, i.e.,

$$2k + 2c_2 + c_4 + \min(c_1, c_3) + r \ge 2j.$$

We further consider three sub-cases: $H(\{n_1, n_3, n_5\}) = H(n_1)$, $H(\{n_1, n_3, n_5\}) = H(n_3)$, and $H(\{n_1, n_3, n_5\}) = H(n_5)$.

For $H(\{n_1, n_3, n_5\}) = H(n_1)$**,** the frequencies in n_3 and n_5 but not duplicated in n_1, n_2 and n_6 must be assigned to calls from m_0, m_1 or m_2. Thus we have $C(\{n_3, n_5\}) - r \le s_1$, which implies that $2i - c_1 - c_3 - r \le 3i + j + 2k$, i.e.,

$$i + 2k + c_1 + c_3 + r \ge -j.$$

As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \ge 3(2k + 2c_2 + c_4 + \min(c_1, c_3) + r) + 2(i + 2k + c_1 + c_3 + r) + 2c_1 + 3(c_1 - \min(c_1, c_3)) + (c_2 + c_3) + 4(c_3 + c_4) + 2r \ge 4j$ because the first term is at least $6j$, the second term is at least $-2j$ and all the other terms are non-negative.

For $H(\{n_1,n_3,n_5\}) = H(n_3)$**,** the frequencies in n_1 and n_5 but not duplicated in n_2, n_3 and n_4 must be assigned to calls from m_4, m_5 or m_6. Thus we have $C(\{n_1,n_5\}) - r \leq s_3$, which implies that $2i - c_3 - r \leq 3i + j + 2k + 2c_1$, i.e.,

$$i + 2k + 2c_1 + c_3 + r \geq -j.$$

As a result, $2i+10k+7\sum_{i=1}^{4} c_i + 7r \geq 3(2k+2c_2+c_4+\min(c_1,c_3)+r)+2(i+2k+2c_1+c_3+r)+3(c_1-\min(c_1,c_3))+(c_2+c_3)+4(c_3+c_4)+2r \geq 4j$ because the first term is at least $6j$, the second term is at least $-2j$ and all the other terms are non-negative.

For $H(\{n_1,n_3,n_5\}) = H(n_5)$**,** the frequencies in n_1 and n_3 but not duplicated in n_4, n_5 and n_6 must be assigned to calls from m_8, m_9 or m_{10}. Thus we have $C(\{n_1,n_3\}) - r \leq s_5$, which implies that $2i - c_1 - r \leq 3i + j + 2k + 2c_3 + \min(c_2,c_4)$, i.e.,

$$i + 2k + c_1 + 2c_3 + \min(c_2,c_4) + r \geq -j.$$

As a result, $2i+10k+7\sum_{i=1}^{4} c_i+7r \geq 3(2k+2c_2+c_4+\min(c_1,c_3)+r)+2(i+2k+c_1+2c_3+\min(c_2,c_4)+r)+4c_1+(c_1+c_2)+2c_4+3(c_3-\min(c_1,c_3))+2(c_4-\min(c_2,c_4))+2r \geq 4j$ because the first term is at least $6j$, the second term is at least $-2j$ and all the other terms are non-negative.

Case 5: $\boldsymbol{H(\{n_1,n_2,\ldots,n_6\}) = H(n_5)}$**.** Similar to the previous cases, we can deduce that $C(\{n_1,n_2,n_3\}) - r \leq s_5 = 3i + j + 2k + 2c_3 + \min(c_2,c_4)$, i.e.,

$$2k + c_1 + 2c_3 + \min(c_2,c_4) + r \geq 0.$$

We further consider three sub-cases: $H(\{n_2,n_4,n_6\}) = H(n_2)$, $H(\{n_2,n_4,n_6\}) = H(n_4)$, and $H(\{n_2,n_4,n_6\}) = H(n_6)$.

For $H(\{n_2,n_4,n_6\}) = H(n_2)$**,** as in the previous cases, we have $i + 2k + c_2 + c_4 + r \geq 2j$. As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \geq 2(i+2k+c_2+c_4+r)+3(2k+c_1+2c_3+\min(c_2,c_4)+r)+4(c_1+c_2)+(c_2+c_3)+3(c_4-\min(c_2,c_4))+2c_4+2r \geq 4j$ because the first term is at least $4j$ and all the other terms are non-negative.

For $H(\{n_2,n_4,n_6\}) = H(n_4)$**,** as in the previous cases, we have $i+2k+2c_2+c_4+\min(c_1,c_3)+r \geq 2j$. As a result, $2i+10k+7\sum_{i=1}^{4} c_i+7r \geq 2(i+2k+2c_2+c_4+\min(c_1,c_3)+r)+3(2k+c_1+2c_3+\min(c_2,c_4)+r)+2(c_1-\min(c_1,c_3))+2c_1+3(c_2-\min(c_2,c_4))+(c_3+c_4)+4c_4+2r \geq 4j$ because the first term is at least $4j$ and all the other terms are non-negative.

For $H(\{n_2,n_4,n_6\}) = H(n_6)$**,** as in the previous cases, we have $i+2k+c_2+2c_4+\min(0,c_3)+r \geq 2j$. As a result, $2i+10k+7\sum_{i=1}^{4} c_i+7r \geq 2(i+2k+c_2+2c_4+\min(0,c_3)+r)+3(2k+c_1+2c_3+\min(c_2,c_4)+r)+2(c_1-\min(0,c_3))+2(c_1+c_2)+3(c_2-\min(c_2,c_4))+(c_3+c_4)+2c_4+2r \geq 4j$ because the first term is at least $4j$ and all the other terms are non-negative.

Case 6: $\boldsymbol{H(\{n_1,n_2,\ldots,n_6\}) = H(n_6)}$**.** Similar to the previous cases, we can deduce that $C(\{n_2,n_3,n_4\}) - r \leq s_6 = 3i + 2k + 2c_4 + \min(0,c_3)$, i.e.,

$$2k + c_1 + c_2 + 2c_4 + \min(0,c_3) + r \geq 2j.$$

We further consider three sub-cases: $H(\{n_1,n_3,n_5\}) = H(n_1)$, $H(\{n_1,n_3,n_5\}) = H(n_3)$, and $H(\{n_1,n_3,n_5\}) = H(n_5)$.

For $H(\{n_1, n_3, n_5\}) = H(n_1)$**,** as in the previous cases, we have $i + 2k + c_1 + c_3 + r \geq -j$.

As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \geq 3(2k + c_1 + c_2 + 2c_4 + \min(0, c_3) + r) + 2(i + 2k + c_1 + c_3 + r) + 2(c_1 + c_2) + 2(c_2 + c_3) + 3(c_3 - \min(0, c_3)) + c_4 + 2r \geq 4j$ because the first term is at least $6j$, the second term is at least $-2j$ and all the other terms are non-negative.

For $H(\{n_1, n_3, n_5\}) = H(n_3)$**,** as in the previous cases, we have $i + 2k + 2c_1 + c_3 + r \geq -j$.

As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \geq 3(2k + c_1 + c_2 + 2c_4 + \min(0, c_3) + r) + 2(i + 2k + 2c_1 + c_3 + r) + 4(c_2 + c_3) + (c_3 + c_4) - 3\min(0, c_3) + 2r \geq 4j$ because the first term is at least $6j$, the second term is at least $-2j$ and all the other terms are non-negative.

For $H(\{n_1, n_3, n_5\}) = H(n_5)$**,** as in the previous cases, we have $i + 2k + c_1 + 2c_3 + \min(c_2, c_4) + r \geq -j$.

As a result, $2i + 10k + 7\sum_{i=1}^{4} c_i + 7r \geq 3(2k + c_1 + c_2 + 2c_4 + \min(0, c_3) + r) + 2(i + 2k + c_1 + 2c_3 + \min(c_2, c_4) + r) + 2(c_1 + c_2) + 2(c_2 - \min(c_2, c_4)) + 3(c_3 - \min(0, c_3)) + c_4 + 2r \geq 4j$ because the first term is at least $6j$, the second term is at least $-2j$ and all the other terms are non-negative.