Online Frequency Allocation in Cellular Networks

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ABSTRACT

Given a mobile telephone network, whose geographical coverage area is divided into cells, phone calls are serviced by assigning frequencies to them, so that no two calls emanating from the same or neighboring cells are assigned the same frequency. Assuming an online arrival of calls and the calls will not terminate, the problem is to minimize the span of frequencies used.

By first considering χ-colorable networks, which is a generalization of (the 3-colorable) cellular networks, we present a $(\chi + 1)/2$-competitive online algorithm. This algorithm, when applied to cellular networks, is effectively a positive solution to the open problem posed in [8]: Does a 2-competitive online algorithm exist for frequency allocation in cellular networks? We further prove a lower bound which shows that our 2-competitive algorithm is optimal.

We discover that an interesting phenomenon occurs for the online frequency allocation problem when the number of calls considered becomes large: previously-derived optimal (lower and upper) bounds on competitive ratios no longer hold true. For cellular networks, we show new asymptotic lower and upper bounds of 1.5 and 1.9126, respectively, which breaks through the optimal bound of 2 shown previously.

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1. INTRODUCTION

Wireless communication based on Frequency Division Multiplexing (FDM) technology is widely used in the area of mobile computing today. In such FDM networks, a geographic area is divided into small cellular regions or *cells*, each containing one base station. Base stations communicate with each other via a high-speed wired network. Calls between any two clients (even within the same cell) must be established through base stations. When a call arrives, the nearest base station must allocate a frequency from the available spectrum to the call without causing any interference to other calls. Interference may occur, which distorts the radio signals, when the same frequency is assigned to two different calls emanating from cells that are geographically close to each other. To avoid interference, the temptation is to use many frequencies. However, frequency spectrum is a scarce resource and thus efficient utilization of the available spectrum is essential for FDM networks.

The *frequency allocation problem* has been extensively studied [1, 3, 4, 10, 12, 13, 15, 16, 20]. Both the off-line and online versions of the problem have been studied. For the off-line problem on cellular networks (where cells are hexagonal regions and the calls to be serviced are known a priori), McDiarmid and Reed [15] have shown that the problem is NP-hard, and 4/3-approximation algorithms were given in [15, 17].

Online Focus. In this paper, we focus on the online version of the frequency allocation problem, in which a sequence σ of calls are arriving over time where $\sigma = (c_1, c_2, \ldots, c_t, \ldots, c_n)$ and c_t represents the cell from which the t-th call emanates. The t-th call is assigned, without information about future calls, i.e., i-th call for $i > t$, a frequency from the integer set $Z^+ = \{1, 2, \ldots\}$ of available frequencies, that is different from that of other calls in the same cell or adjacent cells. Let $f_t \in Z^+$ denote the integer frequency assigned to the t-th call. In other words, $f_t \neq f_i$ for $i < t$, and $c_i = c_t$ or c_i is adjacent to c_t. Assuming that calls never disappear and the integer frequency once assigned to a call cannot be changed, the *online frequency allocation problem* is to minimize the span of frequencies assigned, i.e., $\max\{f_i - f_j \mid i, j \in \{1, \ldots, n\}\}$.

We focus on the online frequency allocation problem for cellular networks (where the cells are hexagonal regions as shown in Figure 1, a conventional model used in wireless communication networks). We call the problem FAC, which stands for frequency allocation in cellular networks.

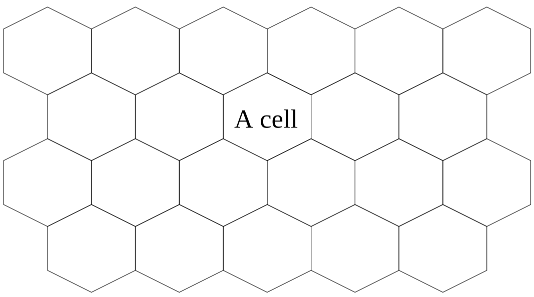

Figure 1: Example of a cellular network (with hexagonal cells).

Performance Measures. We use competitive analysis [2] to measure the performance of our online algorithms. For any given sequence σ of calls, let $\mathcal{A}(\sigma)$ denote the cost of an online algorithm $\mathcal{A}$, i.e., the span of frequencies used by the algorithm $\mathcal{A}$, and let $\mathcal{O}(\sigma)$ denote the cost of the optimal off-line algorithm which knows the whole sequence in advance. The *(absolute) competitive ratio* of algorithm $\mathcal{A}$ is defined to be $R_{\mathcal{A}} = \sup_{\sigma} \mathcal{A}(\sigma)/\mathcal{O}(\sigma)$. Meanwhile, when the number of calls emanating from each cell is large, the *asymptotic competitive ratio* of algorithm $\mathcal{A}$, which is also a concern in this paper, is defined to be

$$R_{\mathcal{A}}^{\infty} = \limsup_{n\to\infty} \max_{\sigma} \left\{ \frac{\mathcal{A}(\sigma)}{\mathcal{O}(\sigma)} \mid \mathcal{O}(\sigma) = n \right\}.$$

Clearly, for any online algorithm $\mathcal{A}$, $R_{\mathcal{A}}^{\infty} \leq R_{\mathcal{A}}$.

Known Results. Previous results have focused on absolute upper bounds. Two simple strategies for frequency allocation have been proposed: *fixed allocation assignment* (FAA) [14] and *greedy algorithm* (Greedy) [8].

FAA partitions cells into independent sets which are each assigned a separate set of frequencies. It is easy to see that FAA for FAC is 3-competitive as cellular networks are 3-colorable.

Greedy assigns the minimum available number (frequency) to a new call so that the call does not interfere with calls of the same or adjacent cells. Caragiannis et al. [8] proved that the competitive ratio of Greedy for FAC is at least 17/7 and at most 2.5. Chan et al. [4] gave a tighter analysis to show that Greedy is 17/7-competitive. Whether there exists a 2-competitive online algorithm for FAC was cited as an open problem in [8].

Our Contributions. In this paper we make the following two main contributions:

Firstly, we present a new general algorithm, called Hybrid, which is a combination of FAA and Greedy. The Hybrid algorithm works for χ-colorable graphs. Though simple and its analysis straightforward, the algorithm solves the open problem posed in [8]. It achieves a competitive ratio of $(\chi + 1)/2$ for χ-colorable graphs, which implies a 2-competitive algorithm for FAC. Then, we give a matching lower bound to show that Hybrid is indeed optimal for FAC.

Secondly, when the number of calls becomes very large, we guarantee better competitive ratios. By generalizing the Hybrid algorithm, we show that it has an asymptotic competitive ratio 1.9126, which is better than the absolute competitive ratio 2. In so doing, we propose an algorithm so as to yield good asymptotic bounds as well as the optimal absolute bounds. Finally, we derive a lower bound 1.5 on the asymptotic competitive ratio for FAC.

Paper Organization. The rest of the paper is organized as follows. In Section 2, we present Hybrid the new online algorithm for FAC, with a competitive ratio of at most 2 and show, by deriving lower bounds that Hybrid is optimal for such networks. In Section 3, we consider large-scale input and asymptotic bounds. Concluding remarks are given in Section 4.

2. ABSOLUTE BOUNDS FOR FAC

In this section we first study the frequency allocation problem for general χ-colorable networks. We introduce the Hybrid algorithm and show that its competitive ratio is at most $(\chi + 1)/2$. When applied to FAC, Hybrid is 2-competitive, given that cellular networks are a special case of 3-colorable networks.

Having obtained upper bounds for FAC, we then study lower bounds. We prove tight lower bound of 2. This allows us to conclude that the Hybrid algorithm is an optimal online algorithm for FAC. As a corollary, (same result as given in [3]), Hybrid is 1.5-competitive, which is optimal, for linear cellular networks given that they are 2-colorable.

2.1 Upper bound

The Hybrid algorithm can be described as follows:

Preprocessing: Given a χ-colorable network, where the nodes have been colored with χ colors, we first partition the frequencies $\{1, 2, \ldots\}$ into $\chi + 1$ disjoint subsets, F_i for

$i = 0, 1, \ldots, \chi$ as follows.

$$
\begin{aligned}
F_0 &= \{1, \chi+2, 2\chi+3, \ldots\} \\
F_1 &= \{2, \chi+3, 2\chi+4, \ldots\} \\
&\vdots \\
F_i &= \{i+1, \chi+i+2, 2\chi+i+3, \ldots\} \\
&\vdots \\
F_\chi &= \{\chi+1, 2\chi+2, 3\chi+3, \ldots\}
\end{aligned}
$$

Frequency Assignment Scheme: For each new call, supposing that it emanates from a node v with color x ($1 \leq x \leq \chi$), we assign a frequency to the call either from F_0 or F_x according to the following scheme:

1. Let y be the smallest number in F_0 such that frequency y is not assigned to any call from v or neighbors of v.
2. Let z be the smallest number in F_x such that frequency z is not assigned to any call from v.
3. Assign frequency $\min\{y, z\}$ to the new call.

The Hybrid algorithm is in fact a hybrid combination of ideas behind FAA and Greedy. Like FAA, Hybrid first partitions the frequencies into sets; however, Hybrid divides the frequencies into $\chi + 1$ sets instead of χ, with a special set F_0 whose frequencies may be used by any call from any node. While calls from a given node are assigned frequencies from a single set in FAA, Hybrid assigns frequencies from two sets F_0 and F_x to calls from a given node with color x. The greedy approach is taken when selecting the particular frequency to assign from the two sets.

For example, consider a cellular network as shown in Fig. 2, in which the cells are labeled from 1 to 21, and also colored by "RGB" scheme, such that no two adjacent cells are labeled with the same color. A *call sequence* $\sigma = (1, 5, 7, 1)$

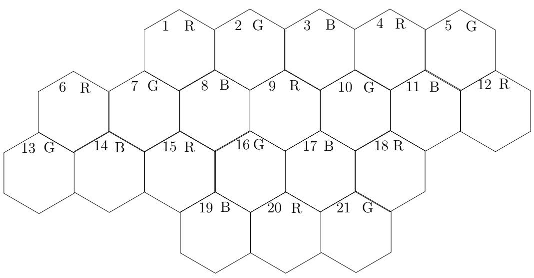

Figure 2: network structure with RGB color scheme.

means calls are emanated from cells 1,5,7,1 respectively in time order. As the time differences between any two subsequent calls can be arbitrarily small or large, simultaneous calls can be represented by sequential calls with arbitrarily small time separation. We use another *frequency sequence* $[1, 2, 1, 3]$ to denote frequencies assigned by an online algorithm to calls of σ, i.e. the online algorithm assigns the frequencies 1, 2, 1, 3 to calls $1, 5, 7, 1$, respectively.

Let us consider a call sequence $\sigma_1 = (8, 1, 6, 10, 8, 4, 8, 19, 6, 8, 13, 11)$. To apply the algorithm Hybrid , we first partition the frequencies into the following sets:

$$
\begin{aligned}
F_0 &= \{1,\ 5,\ 9, \ldots, 4k+1\} \\
F_1 &= \{2,\ 6, 10, \ldots, 4k+2\} \\
F_2 &= \{3,\ 7, 11, \ldots, 4k+3\} \\
F_3 &= \{4,\ 8, 12, \ldots, 4k+4\}
\end{aligned}
$$

F_0 is the shared frequency set, which can be used by any cell. F_1 can be only used in cells with color R, F_2 only used in cells with color G and F_3 only used in cells with color B.

Now apply the algorithm Hybrid to the above mentioned instances σ_1. The array of frequencies assigned by Hybrid is $[1, 2, 1, 1, 4, 2, 5, 1, 2, 8, 3, 4]$, as shown in Figure 3.

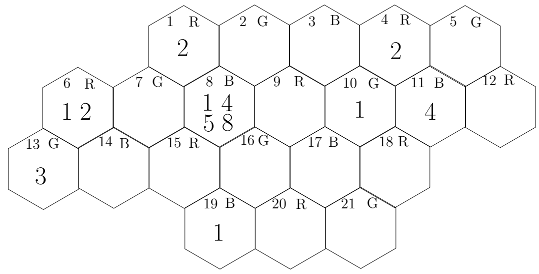

Figure 3: Hybrid works on σ_1

Theorem 1. *The competitive ratio of the* Hybrid *algorithm for χ-colorable networks is $(\chi + 1)/2$.*

Proof. Let h be the highest frequency used by Hybrid on a χ-colorable network. Suppose frequency h is assigned to a call C from node v with color x. When call C was considered by Hybrid, there were two frequencies considered:

1. $y = 1 + i(\chi + 1) \in F_0$ for some integer $i \geq 0$ which implies that there were i calls emanating from node v or neighbors of node v that had already been assigned a frequency from F_0. Either all i of these calls emanate from node v, or there exists a maximum number q such that $0 \leq q \leq i$ and frequency $1 + q(\chi + 1) \in F_0$ is assigned to a particular neighbor of node v, say v' colored x'. In the latter case, we can conclude that there are at least i calls emanating from node v and the neighbor v' since:
   (a) the number of calls from v' assigned a frequency from $F_{x'}$ is at least q (namely, those calls assigned frequencies spanning $(x' + 1)$ to $(x' + 1) + (q - 1)(\chi + 1)$); and
   (b) the number of calls from v' assigned a frequency from F_0 is at least 1 (namely, the call assigned frequency $1 + q(\chi + 1)$); and
   (c) the number of calls from v assigned a frequency from F_0 is at least $i - q - 1$ (namely, those calls assigned frequencies spanning $1 + (q + 1)(\chi + 1)$ to $1 + (i - 1)(\chi + 1)$).
2. $z = (x+1) + j(\chi+1) \in F_x$ for some integer $j \geq 0$ which implies that there were j calls emanating from node v that had already been assigned a frequency from F_x.

Including call C, we can conclude that there are at least $i+j+1$ calls (from F_0, F_x and $F_{x'}$) emanating from node v and the neighbor v' (colored x') of v.

There are two cases to consider:

- Case 1: if $h \in F_0$ then $y < z$ or $1 + i(\chi + 1) < (x + 1) + j(\chi + 1)$ or $j \geq i$
- Case 2: if $h \in F_x$ then $z < y$ or $(x + 1) + j(\chi + 1) < 1 + i(\chi + 1)$ or $i \geq j + 1$

For Case 1, the argument proceeds as follows:

1. Since $j \geq i$, there are at least $2i + 1$ calls emanating from node v or its neighbor v'.
2. If there are at least $2i+1$ calls emanating from node v or its neighbor v', any optimal off-line frequency assignment must use a span of frequencies at least $2i+1$ to avoid interference.
3. The competitive ratio is therefore at most $h/(2i+1) = (1+i(\chi+1))/(2i+1) \leq (\chi+1)/2$ because $(\chi+1) \geq 2$.

For Case 2, the argument proceeds as follows:

1. Since $i \geq j+1$, there are at least $2j+2$ calls emanating from node v or its neighbor v'.
2. If there are at least $2j+2$ calls emanating from node v or its neighbor v', any optimal off-line frequency assignment must use a span of frequencies at least $2j+2$ to avoid interference.
3. The competitive ratio is therefore at most $h/(2j+2) = ((x+1)+j(\chi+1))/(2j+2) \leq (\chi+1)/2$ because $x \leq \chi$.

In both cases, the competitive ratio is at most $(\chi+1)/2$. □

Since the cellular networks in FAC are 3-colorable, we have the following corollary.

Corollary 1. *The competitive ratio of* Hybrid *for FAC is* 2.

If the network is formed by a chain of cells, which is 2-colorable, Hybrid is equivalent to the algorithm proposed in [3] and achieves a competitive ratio 1.5.

Corollary 2. *The competitive ratio of* Hybrid *for a (2-colorable) linear cellular network is* 1.5.

2.2 Lower bound

In this section we show that Hybrid is optimal for FAC by giving a matching lower bound. Precisely, we construct (using an adversary) a problem instance in which no online algorithm can achieve a competitive ratio less than 2.

Theorem 2. *No online algorithm for FAC has a competitive ratio less than* 2.

Proof. Given any online algorithm $\mathcal{A}$ for FAC, consider a cellular network with cells labeled $a, b, c, d, e, f, g, h, i, j$ and k as shown in Figure 4.

The adversary runs as follows. In the first step, calls are made from cells a, b, j and k. Algorithm $\mathcal{A}$ must assign the same single frequency, say 1, to all of these calls; otherwise, the adversary stops and the competitive ratio of $\mathcal{A}$ will be at least 2.

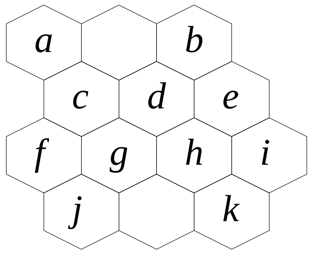

Figure 4: A cellular network for proving the lower bound for FAC.

In the second step, a new call is made from each of cells c and e. If algorithm $\mathcal{A}$ assigns the same frequency to both calls, without loss of generality, say 2, then the adversary will cause a new call from each of cells g and h. It is easy to see that these two new calls require two new frequencies, and thus a total of 4 different frequencies is used by $\mathcal{A}$. Given that the optimal off-line algorithm needs 2 different frequencies, $\mathcal{A}$'s competitive ratio is then 2.

However, if algorithm $\mathcal{A}$ assigns different frequencies to calls from cells c and e, say 2 and 3, the adversary will proceed to make a new call from each of cells f and i. Algorithm $\mathcal{A}$ must then assign 3 to the call from cell f and 2 to the call from cell i; otherwise, the adversary will stop with algorithm $\mathcal{A}$ having used at least 4 frequencies when only 2 were needed. The adversary will continue with a new call from each of cells d, g and h, whereupon algorithm $\mathcal{A}$ must assign three new frequencies to the three new calls. By now, algorithm $\mathcal{A}$ has used 6 frequencies. Given that the optimal off-line algorithm would use only 3, $\mathcal{A}$'s competitive ratio is again 2.

Therefore, at best, $\mathcal{A}$'s competitive ratio is 2. □

3. ASYMPTOTIC BOUNDS FOR FAC

In Section 2 we give an algorithm that achieves a competitive ratio of 2 for FAC and show that no algorithm can achieve a competitive ratio less than 2. So it would seem that FAC problem is completely solved and no further study is necessary.

However, when more and more calls per cell are made: the (asymptotic) bounds for the competitive ratio can be improved to fall below the absolute bound of 2. This is a phenomenon that does exist for some problems (e.g., priority list online scheduling for n bounded size independent jobs on 2 and 3 machines, for which the competitive ratio is tight at 3/2 and 4/3, respectively, when n is small but approaches 1 when n is large).

To understand this phenomenon for the frequency allocation problem, we need to re-examine the lower bound proofs. For example, in the lower bound proof for FAC, the adversary creates a worst case scenario in which the optimal off-line algorithm requires only 2 or 3 frequencies, while any online algorithm is forced to use at least 4 or 6 frequencies, giving therefore a competitive ratio of 2. The critical step occurs when the online algorithm is presented with one call emanating from each of two non-neighboring cells (separated by two cells) arriving at the same time. The algorithm makes a decision whether these two calls will be assigned the same new frequency or different new frequencies. Choosing different frequencies would mean that at least 2 frequencies are used and it becomes clear that the online algorithm would make the wrong choice if the adversary stopped further calls because the optimal solution would only use one frequency. The competitive ratio is 2. On the other hand, choosing the same frequency would compel the adversary to continue to make more calls in order to prove the online algorithm's choice was flawed and a competitive ratio of 2. Note that this choice of whether to assign the same or different frequencies to two calls is a discrete one. However, when dealing with a large number of calls per cell (instead of one call per non-neighboring cell), the choice can be less discrete. For example, we can choose to assign the same frequency to half of the calls and a different frequency to the other half of the calls to the dismay of the adversary. It is this concept that allows a breakthrough on the lower bound.

As it turns out, we can modify the HYBRID algorithm to achieve better performance when dealing with a large number of calls. Recall that HYBRID is a hybrid combination of the Fixed Allocation Approach (FAA) and the greedy (Greedy) approach, with the spectrum of frequencies divided into 4 sets, one of which is used by cells of all the three colors. This set we call the *shared* set. The observation is that, when all calls emanate from a single isolated cell, the Greedy approach gives the optimal solution. So ideally, we would want the Greedy component to play a bigger role when the number of calls is large. This is possible if we increase the size of the shared set. The question is: what is the right size for the shared set? Hence, our new algorithm has effectively two parameters, the standard size of the sets and the size of the larger shared set. The appropriate sizes to achieve the best upper bound are derived.

3.1 Asymptotic Upper Bound

We propose a family of HYBRID algorithms characterized by two integer parameters α and β. The two parameters enable us to control the "degree" to which we combine the fixed allocation and greedy approaches. In the extreme case, when $\alpha = 0$, HYBRID becomes a pure fixed allocation algorithm. On the other hand, when $\beta = 0$, HYBRID becomes a pure greedy algorithm. The particular HYBRID algorithm in Section 2 has $\alpha = 1$ and $\beta = 1$. In the following we give a general description of the family of HYBRID algorithms.

Let $\Delta = \alpha + 3\beta$. Conceptually, frequencies are divided into groups of Δ frequencies. A frequency f is in group i if $i\Delta < f \leq (i+1)\Delta$ for $i \geq 0$. The online algorithm partitions the set of all frequencies $\{1, 2, \ldots\}$ into four disjoint subsets, F_0, F_1, F_2 and F_3. Subset F_0 receives α frequencies from each group while each of F_1, F_2 and F_3 receives β. Since we are focusing on the asymptotic behavior of the online algorithms, as long as the proportion of frequencies from each group among the subsets are fixed, the exact distribution of the Δ frequencies from each group to these four subsets does not affect the overall performance of the algorithm. For instance, consider the following particular distribution of frequencies from group i to the four subsets. Let $\gamma = \min\{\alpha, \beta\}$.

$$
\begin{aligned}
F_0 &= \{i\Delta+1, i\Delta+5, \ldots, i\Delta+4\gamma-3\} \cup \\
&\quad \{i\Delta+4\gamma+j \mid 1 \leq j \leq \alpha-\beta\} \\
F_1 &= \{i\Delta+2, i\Delta+6, \ldots, i\Delta+4\gamma-2\} \cup \\
&\quad \{i\Delta+4\gamma-2+3j \mid 1 \leq j \leq \beta-\alpha\} \\
F_2 &= \{i\Delta+3, i\Delta+7, \ldots, i\Delta+4\gamma-1\} \cup \\
&\quad \{i\Delta+4\gamma-1+3j \mid 1 \leq j \leq \beta-\alpha\} \\
F_3 &= \{i\Delta+4, i\Delta+8, \ldots, i\Delta+4\gamma\} \cup \\
&\quad \{i\Delta+4\gamma+3j \mid 1 \leq j \leq \beta-\alpha\}
\end{aligned}
$$

The family of HYBRID algorithms assigns a frequency for a new call using the same frequency assignment scheme as in Section 2.1.

There is a simple property of HYBRID which is useful for analysis.

LEMMA 3. *If a frequency of group k is assigned by a cell colored c, then the number of frequencies of F_c assigned by the cell is at least βk.*

PROOF. All the frequencies of F_c in group i for $0 \leq i \leq k-1$, which are lower than any frequency in group k, must have all been assigned to calls from the cell. Since there are βk such frequencies, the lemma follows. □

We show that the asymptotic competitive ratio of HYBRID approaches 1.9126 when β/α approaches 0.8393. First, we have the following lemma to lower bound the minimum number of frequencies required by the optimal off-line algorithm, which is the total number of calls emanating from three mutually adjacent cells.

LEMMA 4. *If a cell A assigns a frequency from group k, then the total number of calls from cell A and two of its neighbors, which are also adjacent to each other, is at least $(\alpha+\beta)k - \beta(1+1/\alpha)$ for $\beta/\alpha \geq -\frac{2}{3} + \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3} (\approx 0.8393)$, and at least $(\beta^3/\alpha^2 + 2\beta^2/\alpha + \beta - \alpha)k - \beta(1+1/\alpha)$ for $\beta/\alpha \leq -\frac{2}{3} + \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3} (\approx 0.8393)$.*

PROOF. Let $B_1, B_2, \ldots, B_6$ denote the six neighbors of cell A in clockwise order as shown in Figure 5. We divide the six neighbors into three groups $\{B_1, B_2\}$, $\{B_3, B_4\}$, and $\{B_5, B_6\}$.

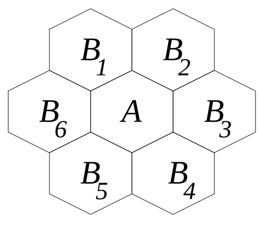

Figure 5: Cell A and its neighboring cells.

Let F' be the subset of F_0 containing all the frequencies from group 0 to group $k-1$, precisely, $F' = \{f \mid f \in F_0 \text{ and } f \leq k\Delta\}$. Thus, $|F'| = \alpha k$. Since a frequency from group k is assigned by cell A, by the algorithm, each frequency in F' should have already been assigned by cell A or its neighbors, cells $B_1, B_2, \ldots, B_6$. Assume that x of them are assigned by cell A, y by cells B_1 and B_2, z by cells B_3 and B_4, and at least $\alpha k - x - y - z$ by cells B_5 or B_6. Without loss of generality, we can assume that the highest frequency assigned from F' by cell B_1 or B_2 is at least that assigned from F' by cell B_3 or B_4, and the highest frequency assigned from F' by cell B_3 or B_4 is at least that assigned from F' by cell B_5 or B_6.

Suppose the color of cell A is c_1, cells B_1, B_3 and B_5 are c_2, and cells B_2, B_4 and B_6 are c_3. Consider the total number of distinct frequencies (calls) from cells A, B_1 and B_2.

- There are βk frequencies of F_{c_1} from groups 0 to $k-1$ assigned by cell A.
- There are $x+y$ frequencies from F' assigned by cells A, B_1 and B_2.
- We claim that there are at least $\beta\lfloor(\alpha k - x - 1)/\alpha\rfloor$ frequencies of F_{c_2} or F_{c_3} assigned by cell B_1 or B_2. We can see that $\alpha k - x$ frequencies in F' are assigned by cells B_1, B_2, B_3, B_4, B_5 and B_6. The highest frequency among these frequencies must be from group $\geq \lfloor(\alpha k - x - 1)/\alpha\rfloor$, and by the assumption the frequency is from B_1 or B_2. Thus by Lemma 3, the claim follows.

Note that all these frequencies are from disjoint sets. Thus the total number of calls from cells A, B_1 and B_2 is at least

$$\begin{aligned} T_1 &= \beta k + x + y + \beta\lfloor(\alpha k - x - 1)/\alpha\rfloor \\ &\geq 2\beta k + (1 - \beta/\alpha)x + y - \beta(1 + 1/\alpha). \end{aligned}$$

Consider the total number of distinct frequencies (calls) from cells A, B_3 and B_4.

- There are βk frequencies of F_{c_1} from groups 0 to $k-1$ assigned by cell A.
- By the assumption, there are $x+z$ frequencies from F' assigned by cells A, B_3 and B_4.
- We claim that there are at least $\beta\lfloor(\alpha k - x - y - 1)/\alpha\rfloor$ frequencies of F_{c_2} or F_{c_3} assigned by cell B_3 or B_4. We can see that $\alpha k - x - y$ frequencies in F' are assigned by cells B_3, B_4, B_5 and B_6. The highest frequency among these frequencies must be from group $\geq \lfloor(\alpha k - x - y - 1)/\alpha\rfloor$, and by the assumption the frequency is from B_3 or B_4. Thus by Lemma 3, the claim follows.

Note that all these frequencies are from disjoint sets. Thus the total number of calls from cells A, B_3 and B_4 is at least

$$\begin{aligned} T_2 &= \beta k + x + z + \beta\lfloor(\alpha k - x - y - 1)/\alpha\rfloor \\ &\geq 2\beta k + (1 - \beta/\alpha)x - (\beta/\alpha)y + z - \beta(1 + 1/\alpha). \end{aligned}$$

Consider the total number of distinct frequencies (calls) from cells A, B_5 and B_6.

- There are at least βk frequencies of F_{c_1} from groups 0 by $k-1$ assigned by cell A.
- By the assumption, there are at least $\alpha k - y - z$ frequencies from F' assigned by cells A, B_5 and B_6.
- We claim that there are at least $\beta\lfloor(\alpha k - x - y - z - 1)/\alpha\rfloor$ frequencies of F_{c_2} or F_{c_3} assigned by cells B_5 or B_6. The claim is true because $\alpha k - x - y - z$ frequencies from F' are assigned by cell B_5 and B_6, the highest of which must be from group $\geq \lfloor(\alpha k - x - y - z - 1)/\alpha\rfloor$. Hence, by Lemma 3, the claim follows.

Note that all these frequencies are from disjoint sets. Thus the total number of calls from cells A, B_5 and B_6 is at least

$$\begin{aligned} T_3 &= \beta k + \alpha k - y - z + \beta\lfloor(\alpha k - x - y - z - 1)/\alpha\rfloor \\ &\geq (2\alpha + \beta)k - (\beta/\alpha)x - (1 + \beta/\alpha)y \\ &\quad -(1 + \beta/\alpha)z - \beta(1 + 1/\alpha). \end{aligned}$$

The total number of calls from cells A and two of its neighbors, which are also adjacent to each other, is bounded below by the value $\max\{T_1, T_2, T_3\}$. It can be verified that when $z = (1 + \beta/\alpha)y$ and $x = \beta k - y(3 + 3\beta/\alpha + \beta^2/\alpha^2)$, $\max\{T_1, T_2, T_3\}$ achieves the minimum value, where the values of T_1, T_2 and T_3 are all equal, which is

$$(\alpha + \beta)k + y(\beta^3/\alpha^3 + 2\beta^2/\alpha^2 - 2) - \beta(1 + 1/\alpha).$$

For $\beta/\alpha \geq -\frac{2}{3} + \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3}$, we have $\beta^3/\alpha^3 + 2\beta^2/\alpha^2 - 2 \geq 0$. Therefore, $\max\{T_1, T_2, T_3\} \geq (\alpha + \beta)k - \beta(1 + 1/\alpha)$. On the other hand, since $y \leq \alpha k$, when $\beta/\alpha \leq -\frac{2}{3} + \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3}$, we have $\beta^3/\alpha^3 + 2\beta^2/\alpha^2 - 2 \leq 0$, and hence $\max\{T_1, T_2, T_3\} \geq (\alpha + \beta)k + \alpha k(\beta^3/\alpha^3 + 2\beta^2/\alpha^2 - 2) - \beta(1 + 1/\alpha) = (\beta^3/\alpha^2 + 2\beta^2/\alpha + \beta - \alpha)k - \beta(1 + 1/\alpha)$. As a result, the lemma follows. □

By the above lemma, if we set the values of α and β such that $\beta/\alpha \geq -\frac{2}{3} + \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3}$, HYBRID has the asymptotic competitive ratio $(\alpha + 3\beta)/(\alpha + \beta)$.

THEOREM 5. *The asymptotic competitive ratio of* HYBRID *for FAC is* $(\alpha+3\beta)/(\alpha+\beta)$ *for* $\beta/\alpha \geq -\frac{2}{3} + \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3}$ (≈ 0.8393).

PROOF. Suppose the highest frequency used by HYBRID is from group k assigned by a cell A, which is at most $(k+1)\Delta = (\alpha + 3\beta)(k + 1)$. By Lemma 4, for $\beta/\alpha \geq -\frac{2}{3} + \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3}$, the total number of calls from cell A and two of its neighbors, which are also adjacent to each other, is at least $(\alpha + \beta)k - \beta(1 + 1/\alpha)$, and hence, the span of frequencies used must be at least $(\alpha + \beta)k - \beta(1 + 1/\alpha)$. Therefore, the asymptotic competitive ratio of HYBRID is

$$R^\infty \leq \lim_{k\to\infty} \frac{(\alpha + 3\beta)(k + 1)}{(\alpha + \beta)k - \beta(1 + 1/\alpha)} = \frac{\alpha + 3\beta}{\alpha + \beta}$$

□

COROLLARY 3. *The asymptotic competitive ratio of* HYBRID *for FAC approaches* $\frac{11}{3} + \frac{2(19-3\sqrt{33})^{1/3}}{9} - \frac{2(19-3\sqrt{33})^{2/3}}{9} + \frac{2(19+3\sqrt{33})^{1/3}}{9} - \frac{2(19+3\sqrt{33})^{2/3}}{9}$ (≈ 1.9126).

PROOF. As we can assign integer values to α and β such that β/α is arbitrarily close to $-\frac{2}{3} + \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3}$, we have $(\alpha+3\beta)/(\alpha+\beta)$ arbitrarily close to $\frac{11}{3} + \frac{2(19-3\sqrt{33})^{1/3}}{9} - \frac{2(19-3\sqrt{33})^{2/3}}{9} + \frac{2(19+3\sqrt{33})^{1/3}}{9} - \frac{2(19+3\sqrt{33})^{2/3}}{9}$. □

In practice, it is preferable to have small values of α and β while the performance can be maintained. For example, we can set the values of α and β to 13 and 11, respectively.

By Theorem 5, this particular Hybrid algorithm has an asymptotic competitive ratio $23/12 \approx 1.9167$. In the following theorem, we also prove that at the same time this algorithm achieves the optimal absolute competitive ratio, i.e., 2, by a similar but more conservative analysis as in Lemma 4.

Theorem 6. *For $\alpha = 13$ and $\beta = 11$, the (absolute) competitive ratio of* Hybrid *for FAC is* 2.

Proof. As we are deriving the absolute competitive ratio of the Hybrid algorithm, we have to fix the distribution of the Δ frequencies of a group to the four subsets. In particular, we follow the distribution stated in Section 3.1, i.e., for frequencies of group k,

$$\begin{aligned} F_0 &= \{46k+1, 46k+5, \ldots, 46k+41\} \\ &\quad \cup\{46k+45, 46k+46\} \\ F_1 &= \{46k+2, 46k+6, \ldots, 46k+42\} \\ F_2 &= \{46k+3, 46k+7, \ldots, 46k+43\} \\ F_3 &= \{46k+4, 46k+8, \ldots, 46k+44\} \end{aligned}$$

Assume that the highest frequency assigned by Hybrid is for a cell A and the frequency is from group $k \geq 1$. If $k = 0$, the scenario is similar to the case for $\alpha = \beta = 1$ where we prove a competitive ratio of 2 in Section 2.1. Let the highest frequency be $h = 46k + 4i - 3 + c + j$ for some integers $1 \leq i \leq 11$ and $j \in \{0, 4, 5\}$ and $0 \leq c \leq 4$. Note that c represents the color to which the frequency belongs (color 0 to represent F_0), and $j = 4$ or 5 if the frequency is either $46k + 45$ or $46k + 46$ that belongs to F_0. The core part of the proof consists of a more conservative analysis than that in Lemma 4 to lower bound the number of calls from cell A and two of its neighbors, and hence show that the competitive ratio of Hybrid is 2.

Similar to the proof of Lemma 4, let $B_1, B_2, \ldots, B_6$ denote the six neighbors of cell A in clockwise order as shown in Figure 5. Suppose the color of cell A is c_1, cells B_1, B_3 and B_5 are c_2, and cells B_2, B_4 and B_6 are c_3. We divide the six neighbors into three groups $\{B_1, B_2\}$, $\{B_3, B_4\}$, and $\{B_5, B_6\}$.

Let F' be a subset of F_0 containing the frequencies of at most h, precisely, $F' = \{f \mid f \in F_0 \text{ and } f \leq h\}$. Thus, $|F'| = \beta k + i + j' = 13k + i + j'$ where $j' = \max\{0, j - 3\}$. By the algorithm, each frequency in F' should have already been assigned by cell A or its neighbors, cells $B_1, B_2, \ldots, B_6$. Assume that x of them are assigned by cell A, y by cells B_1 and B_2, z by cells B_3 and B_4, and at least $13k+i+j'-x-y-z$ to cells B_5 or B_6. Without loss of generality, we can assume that the highest frequency assigned from F' by cell B_1 or B_2 is at least that assigned from F' by cell B_3 or B_4, and the highest frequency assigned from F' by cell B_3 or B_4 is at least that assigned by cell B_5 or B_6.

Consider the total number of distinct frequencies (calls) from cells A, B_1 and B_2.

- There are at least $11k+i-c'$ frequencies of F_{c_1} assigned by cell A, where $c' = 1$ if $c = 0$, i.e., the highest frequency assigned is from F_0, and $c' = 0$ if otherwise.
- By the assumption, there are $x + y$ frequencies from F' assigned by cells A, B_1 and B_2.
- It can be verified that there are at least $11(13k + i + j' - x - 1)/13$ frequencies of F_{c_2} or F_{c_3} assigned by cell B_1 or B_2.

Note that all these frequencies are from disjoint sets. Thus the total number of calls from cells A, B_1 and B_2 is at least

$$\begin{aligned} T_1 &= 11k + i - c' + x + y + 11(13k + i + j' - x - 1)/13 \\ &= 22k + 24i/13 + 11j'/13 + 2x/13 + y - c' - 11/13. \end{aligned}$$

Consider the total number of distinct frequencies (calls) from cells A, B_3 and B_4.

- There are $11k + i - c'$ frequencies of F_{c_1} assigned by cell A.
- By assumption, there are $x + z$ frequencies from F' assigned by cells A, B_3 and B_4.
- It can be verified that there are at least $11(13k + i + j' - x - y - 1)/13$ frequencies of F_{c_2} or F_{c_3} assigned by cell B_3 or B_4.

Note that all these frequencies are from disjoint sets. Thus the total number of calls from cells A, B_3 and B_4 is at least

$$\begin{aligned} T_2 &= 11k + i - c' + x + z \\ &\quad +11(13k + i + j' - x - y - 1)/13 \\ &= 22k + 24i/13 + 11j'/13 + 2x/13 \\ &\quad -11y/13 + z - c' - 11/13. \end{aligned}$$

Consider the total number of distinct frequencies (calls) from cells A, B_5 and B_6.

- There are at least $11k+i-c'$ frequencies of F_{c_1} assigned by cell A.
- By assumption, there are at least $x + 13k + i + j' - x - y - z$ frequencies from F' assigned by cells A, B_5 and B_6.
- It can be verified that there are at least $11(13k + i + j' - x - y - z - 1)/13$ frequencies of F_{c_2} or F_{c_3} assigned by cells B_5 or B_6.

Note that all these frequencies are from disjoint sets. Thus the total number of calls from cells A, B_5 and B_6 is at least

$$\begin{aligned} T_3 &= 11k + i - c' + 13k + i + j - y - z + \\ &\quad 11(13k + i + j' - x - y - z - 1)/13 \\ &= 35k + 37i/13 + 24j'/13 - 11x/13 \\ &\quad -24y/13 - 24z/13 - c' - 11/13. \end{aligned}$$

The total number of calls from cells A and two of its neighbors, which are also adjacent to each other, is bounded below by the value $\max\{T_1, T_2, T_3\}$. It can be verified that when $z = 24y/13$ and $x = 13k+i+j-1057y/169$, $\max\{T_1, T_2, T_3\}$ achieves the minimum value, which is

$$\begin{aligned} &24k + 2i + j' + 83y/2197 - c' - 11/13 \\ \geq\ &24k + 2i + j' - c' - 11/13. \end{aligned}$$

As a result the competitive ratio of Hybrid is at most

$$\begin{aligned} &\frac{46k + 4i - 3 + c + j}{24k + 2i + j' - c' - 11/13} \\ \leq\ &\frac{46(k-1) + 46 + 4i + j - (3 - c)}{24(k-1) + 23 + 2i + \max\{0, j-3\} - \max\{0, 1-c\}} \\ \leq\ &2. \end{aligned}$$

(because $k \geq 1$, $j \in \{0, 4, 5\}$, $c \in \{0, 1, 2, 3\}$ and $c' = \max\{0, 1 - c\}$). $\square$

3.2 Asymptotic Lower Bound

We give a lower bound of 1.5 on the asymptotic competitive ratio for FAC.

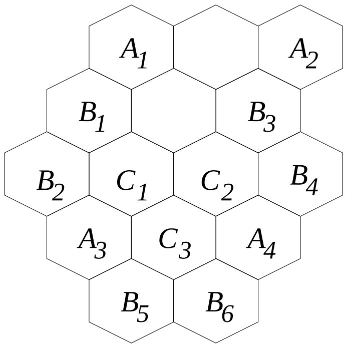

Figure 6: A cellular network for proving the lower bound for FAC.

THEOREM 7. *No online algorithm for FAC has an asymptotic competitive ratio less than* $3/2$.

PROOF. Given any online algorithm $\mathcal{A}$ for FAC, consider the cellular network shown in Figure 6. The adversary consists of three steps.

In step 1, the adversary makes n calls in cells with label A_j $(j = 1, 2, 3, 4)$. Let xn be the minimal number of common frequencies in any two cells labeled with A_j. Then the online algorithm uses at least $(2-x)n$ distinct frequencies while the optimal off-line algorithm uses at least n distinct frequencies. Therefore, the asymptotic competitive ratio in this step is at least $2 - x$. The adversary stops the sequence of calls if $x \leq 1/2$.

Otherwise, in step 2, the adversary makes n calls in each cell with label B_j $(j = \{1, \dots, 6\})$. Consider the four cells A_1, B_1, B_2, A_3. The number of distinct frequencies used by the algorithm $\mathcal{A}$ is at least $(2+x)n$. We assume that in this step, there are a total of $(2 + x + y)n$ frequencies used by the algorithm $\mathcal{A}$. The adversary stops the sequence of calls if $x + y \geq 1$, which implies an asymptotic competitive ratio of at least $3/2$.

Otherwise, in step 3, the adversary makes n calls in each cell with label C_j $(j = 1, 2, 3)$. It is worth noting that the total distinct frequencies used by cells B_1, B_2, A_3 is at least $(2 + x)n$, since B_1 reuses at most $(1 - x)n$ frequencies from A_3 and the frequencies used by B_1 should be totally disjoint from frequencies used by cell B_2. Thus, cell C_1 reuses at most $(2 + x + y)n - (2 + x)n = yn$ old frequencies. This ensures that C_1 has to use at least $(1 - y)n$ new distinct frequencies. One can check that both cells C_2 and C_3 will contribute another $2(1-y)n$ new distinct frequencies. Thus, there are totally $(2+x+y)n+3(1-y)n$ distinct frequencies used by algorithm $\mathcal{A}$. However, the optimal algorithm uses at most $3n$. Hence, $R^{\infty}_{\mathcal{A}} \geq (5 + x - 2y)/3$. Since $y < 1/2$ and $x > 1/2$, we get $R^{\infty}_{\mathcal{A}} \geq 3/2$. □

4. CONCLUDING REMARKS

Many interesting problems arise which are related to the online frequency allocation problem. The frequency allocation problem has been recognized as a variant of the graph multicoloring problem [16], in which each node may be assigned more than one colors and adjacent nodes must have no overlapping colors. One interesting direction is the frequency allocation problem without the assumption that all calls have infinite duration [11].

Since bandwidth is always a valuable resource, one approach to minimize the span of frequencies used is by reassigning new frequencies to some of the existing calls. Thus the problem is to design algorithms with reassignments to achieve lower bandwidth of frequencies when only local information of the neighboring cells is available [7, 11, 19].

In the third generation (3G) mobile communication, Orthogonal Variable Spreading Factor (OVSF) code assignment is a fundamental problem in Wideband Code-Division Multiple-Access (W-CDMA) systems. In the OVSF problem, codes must be assigned to incoming call requests with different data rate requirements from a complete binary code tree, which is more general than frequency allocation. This problem is much more related to real practice. Erlebach et al [9] gave the first h-competitive algorithm which minimizes the number of code assignments/reassignments, where h is the height of the code tree. With the help of extra bandwidth, Chin et al [5] gave a 5-competitive algorithm, furthermore, they proposed a constant competitive algorithm [6] with no resource augmentation. There still exists a big gap between the lower and upper bound of the competitive ratio. How to close the gap remains an interesting problem for further research.

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6. REFERENCES

[1] K. I. Aardal, S. P. M. van Hoesel, A. M. C. A. Koster, C. Mannino, and A. Sassano. Models and solution techniques for frequency assignment problems. *Quarterly Journal of the Belgian, French and Italian Operations Research Societies (4OR)*, 1(4):261–317, 2003.

[2] A. Borodin and R. El-Yaniv. *Online Computation and Competitive Analysis*. Cambridge University Press, 1998.

[3] W.-T. Chan, F. Y. L. Chin, D. Ye, Y. Zhang and H. Zhu. Frequency Allocation Problem for Linear Cellular Networks. Proceedings of the 17th Annual International Symposium on Algorithms and Computation (ISAAC), pages 61-70, Dec 2006.

[4] W.-T. Chan, F. Y. L. Chin, D. Ye, Y. Zhang and H. Zhu. Greedy Online Frequency Allocation in Cellular Networks. *Information Processing Letters*, 102(2007), 55-61.

[5] F. Y. L. Chin, Y. Zhang and H. Zhu. Online OVSF Code Assignment with Resource Augmentation. to appear in the third International Conference on Algorithmic Aspects in Information and Management (AAIM 07).

[6] F. Y. L. Chin, Y. Zhang and H. Zhu. Constant-Competitive Algorithms for Online OVSF Code Assignment. manuscript.

[7] F. Y. L. Chin, Y. Zhang and H. Zhu. A 1-local 4/3-competitive Algorithm for Multicoloring

Triangle-free Hexagonal Graphs. manuscript.

[8] I. Caragiannis, C. Kaklamanis, and E. Papaioannou. Efficient on-line frequency allocation and call control in cellular networks. *Theory Comput. Syst.*, 35(5):521–543, 2002. A preliminary version of the paper appeared in SPAA 2000.

[9] T. Erlebach, R. Jacob, M. Mihalak, M. Nunkesser, G. Szabo and P. Widmayer. An algorithmic view on OVSF code assignment. In proc. of 21-st Symposium on Theoretical Aspects of Computer Science (STACS 2004), LNCS 2996, pp. 270-281.

[10] W. Hale. Frequency assignment: Theory and applications. *Proceedings of the IEEE*, 68(12):1497–1514, 1980.

[11] J. Janssen, D. Krizanc, L. Narayanan, and S. M. Shende. Distributed online frequency assignment in cellular networks. *J. Algorithms*, 36(2):119–151, 2000.

[12] B. Jaumard, O. Marcotte, and C. Meyer. Mathematical models and exact methods for channel assignment in cellular networks. In B. Sansò and P. Soriano, editors, *Telecommunications Network Planning*, pages 239–255. Kluwer, 1999.

[13] I. Katzela and M. Naghshineh. Channel assignment schemes for cellular mobile telecommunication systems: A comprehensive survey. *IEEE Personal Communications*, 3(3):10–31, 1996.

[14] V. MacDonald. Advanced mobile phone service: The cellular concept. *Bell Systems Technical Journal*, 58(1), 1979.

[15] C. McDiarmid and B. A. Reed. Channel assignment and weighted coloring. *Networks*, 36(2):114–117, 2000.

[16] L. Narayanan. Channel assignment and graph multicoloring. In I. Stojmenović, editor, *Handbook of Wireless Networks and Mobile Computing*, pages 71–94. John Wiley & Sons, 2002.

[17] L. Narayanan and S. M. Shende. Static frequency assignment in cellular networks. *Algorithmica*, 29(3):396–409, 2001.

[18] L. Narayanan and Y. Tang. Worst-case analysis of a dynamic channel assignment strategy. *Discrete Applied Mathematics*, 140(1-3):115–141, 2004.

[19] Petra Sparl and Janez Zerovnik. 2-local 4/3-competitive algorithm for multicoloring hexagonal graphs. *J. Algorithms* 55(1): 29-41, 2005.

[20] Yong Zhang. Frequency Allocation in Cellular Networks. Ph.D. Thesis, Fudan University, 2006.