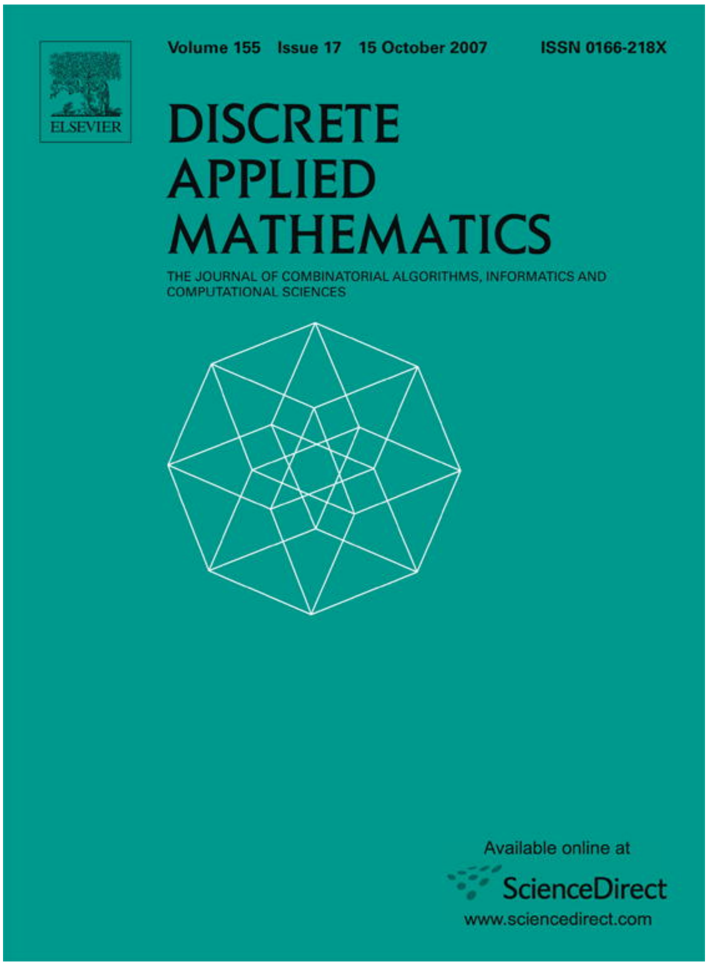
Volume 155 Issue 17 15 October 2007 ISSN 0166-218X
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Discrete Applied Mathematics 155 (2007) 2312 – 2320
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Hamiltonicity of regular graphs and blocks of consecutive ones in symmetric matrices

Rui Wang[a,b], Francis C.M. Lau[a,*], Yingchao Zhao[c]

[a] *Department of Computer Science, The University of Hong Kong, Hong Kong*
[b] *Department of Computer Science, The Ocean University of China, P.R. China*
[c] *Institute for Theoretical Computer Science, Tsinghua University, P.R. China*

Received 22 February 2006; received in revised form 16 November 2006; accepted 11 June 2007
Available online 15 June 2007

Abstract

We show that the Hamiltonicity of a regular graph G can be fully characterized by the numbers of blocks of consecutive ones in the binary matrix $\mathbb{A} + \mathbb{I}$, where $\mathbb{A}$ is the adjacency matrix of G, $\mathbb{I}$ is the unit matrix, and the blocks can be either linear or circular. Concretely, a k-regular graph G with girth $g(G) \geqslant 5$ has a Hamiltonian circuit if and only if the matrix $\mathbb{A} + \mathbb{I}$ can be permuted on rows such that each column has at most (or exactly) $k - 1$ circular blocks of consecutive ones; and if the graph G is k-regular except for two $(k - 1)$-degree vertices a and b, then there is a Hamiltonian path from a to b if and only if the matrix $\mathbb{A} + \mathbb{I}$ can be permuted on rows to have at most (or exactly) $k - 1$ linear blocks per column.

Then we turn to the problem of determining whether a given matrix can have at most k blocks of consecutive ones per column by some row permutation. For this problem, Booth and Lueker gave a linear algorithm for $k = 1$ [Proceedings of the Seventh Annual ACM Symposium on Theory of Computing, 1975, pp. 255–265]; Flammini et al. showed its NP-completeness for general k [Algorithmica 16 (1996) 549–568]; and Goldberg et al. proved the same for every fixed $k \geqslant 2$ [J. Comput. Biol. 2 (1) (1995) 139–152]. In this paper, we strengthen their result by proving that the problem remains NP-complete for every constant $k \geqslant 2$ even if the matrix is restricted to (1) symmetric, or (2) having at most three blocks per row.

Keywords: Regular graph; Hamiltonicity; Consecutive ones; NP-completeness

1. Introduction

Graph Hamiltonicity is an important topic in graph theory and has been studied extensively. Many sufficient and/or necessary conditions have been established for Hamiltonian path or circuit in general graphs as well as in special classes of graphs [1,8,11]. In this paper we study the class of k-regular graphs and present a sufficient and necessary condition for their Hamiltonicity. The condition is stated in terms of "blocks of consecutive ones" in some binary matrices related to the graphs.

An $m \times n$ matrix $\mathbb{M}_{m\times n}$ is *binary* if every entry $\mathbb{M}(i, j)$ is either 1 or 0. Given a binary matrix $\mathbb{M}_{m\times n}$, a *block of consecutive ones* in column j is a maximal interval of consecutive entries $\mathbb{M}(i, j), \mathbb{M}(i + 1, j), \ldots, \mathbb{M}(k, j)$ that are all ones. Obviously, the *block number* of column j is the number of entries $\mathbb{M}(i, j)$ such that $\mathbb{M}(i, j) = 1$ and either

* Corresponding author. Fax: +852 2858 4141.
E-mail addresses: rwang@cs.hku.hk (R. Wang), fcmlau@cs.hku.hk (F.C.M. Lau), yczhao@mails.tsinghua.edu.cn (Y. Zhao).

0166-218X/$ - see front matter
doi:10.1016/j.dam.2007.06.009

$\mathbb{M}(i+1, j)=0$ or $i=m$. Such blocks are called *linear* blocks as they are defined with the assumption that the first row is non-adjacent to the last row. If we consider the first row and the last row to be adjacent we have *circular* blocks. The number of circular blocks in column j of a binary matrix $\mathbb{M}_{m\times n}$ is the number of entries $\mathbb{M}(i, j)$ such that $\mathbb{M}(i, j)=1$ and either $\mathbb{M}(i+1, j)=0$ or $i=m$ and $\mathbb{M}(1, j)=0$.

A graph $G=(V, E)$ with $V=\{v_1, v_2, \ldots, v_n\}$ is *k-regular* if every vertex $v \in V$ has a degree $d(v)=k$. The girth of G, denoted by $g(G)$, is the length of a minimum circuit (cycle) in G. The adjacency matrix of G, denoted by $\mathbb{A}(G)$, is an $n \times n$ symmetric binary matrix such that $\mathbb{A}(i, j)=1$ if and only if $(v_i, v_j) \in E$. The unit matrix, denoted by $\mathbb{I}$, is an $n \times n$ binary matrix such that $\mathbb{I}(i, j)=1$ if and only if $i=j$.

Our main results on the Hamiltonicity, given in Section 2, are (1) a k-regular graph G with $k \geqslant 3$ and $g(G) \geqslant 5$ contains a Hamiltonian circuit if and only if the rows of the matrix $\mathbb{A}(G)+\mathbb{I}$ can be permuted to make every column having at most $k-1$ circular blocks of consecutive ones, and (2) if G is k-regular except for two vertices a and b which are of degree $k-1$, then G contains a Hamiltonian path between a and b if and only if the rows of the matrix $\mathbb{A}(G)+\mathbb{I}$ can be permuted to make every column having at most $k-1$ linear blocks. Moreover, these two results still hold if "at most" is replaced by "exactly".

The problem of consecutive ones blocks has been shown to be useful in such areas as data compression, information retrieval, interval graphs, DNA computing, and interval routing [9,7,3,12]. Many variants on the total number of blocks over all columns have been shown to be NP-complete (see for instance A4 in [4]). For the maximum number of blocks in each column, Gavoille and Peleg [6] gave an upper bound $m/4+(1/4)\sqrt{2m\ln mn}$ for $m \times n$ matrices. For the problem to determine whether a given matrix can be row permuted to have at most k blocks per column, Booth and Lueker [2] presented a linear algorithm for $k=1$; Flammini et al. [3] showed that the problem is NP-complete, and Goldberg et al. [7] proved the same for every constant $k \geqslant 2$. In Section 3, we strengthen this last result by showing that for every fixed $k \geqslant 2$ the problem remains NP-complete even if restricted to symmetric matrices or matrices having at most three blocks per row.

For convenience, in the following, "blocks" means (linear or circular) blocks of consecutive ones.

2. Hamiltonicity of regular graphs

In this section, we establish a connection between Hamiltonian circuit and circular blocks and a connection between Hamiltonian path and linear blocks.

Theorem 2.1. *Any k-regular graph* $G=(V, E)$ *with* $g(G) \geqslant 5$ *has a Hamiltonian circuit if and only if the matrix* $\mathbb{A}(G)+\mathbb{I}$ *can be row permuted to have at most* $k-1$ *circular blocks in each column.*

Proof. Let $V=\{v_1, v_2, \ldots, v_n\}$ and $\mathbb{B}=\mathbb{A}(G)+\mathbb{I}$. If G has a Hamiltonian circuit, $v_{i_1}v_{i_2}\ldots v_{i_n}v_{i_1}$, consider the permutation that arranges the rows of $\mathbb{B}$ in the order of $i_1i_2\ldots i_n$. With this permutation, for any column i_j, the adjacency of row i_{j-1} (or i_n if $i_j=1$) and row i_j and the adjacency of row i_j and row i_{j+1} (or i_1 if $i_j=n$) will make the three 1-entries, $\mathbb{B}(i_{j-1}, i_j)$, $\mathbb{B}(i_j, i_j)$, and $\mathbb{B}(i_{j+1}, i_j)$, consecutive in column i_j. So, column i_j has no more than $k-1$ blocks (because each column has exactly $k+1$ ones). The *only if* part of the lemma is proved.

To prove the *if* part, we need to consider what will happen after making two rows, i_1 and i_2, adjacent. If v_{i_1} is adjacent to v_{i_2} in G, then making row i_1 and row i_2 adjacent in $\mathbb{B}$ makes the two 1-entries, $\mathbb{B}(i_1, i_1)$ and $\mathbb{B}(i_2, i_1)$, adjacent in column i_1 and the two 1-entries, $\mathbb{B}(i_1, i_2)$ and $\mathbb{B}(i_2, i_2)$, adjacent in column i_2. But for any column i_3 other than i_1 and i_2, we have either $\mathbb{B}(i_1, i_3)=0$ or $\mathbb{B}(i_2, i_3)=0$ (otherwise G has a circuit $v_{i_1}v_{i_2}v_{i_3}v_{i_1}$ of length 3, contradicting $g(G) \geqslant 5$). So, if v_{i_1} is adjacent to v_{i_2} in G, then making row i_1 and row i_2 adjacent in $\mathbb{B}$ makes exactly two pairs of 1's adjacent in all the columns (one pair in column i_1 and the other in column i_2).

If v_{i_1} is not adjacent to v_{i_2}, then for any two columns, j_1 and j_2, the four entries, $\mathbb{B}(i_1, j_1)$, $\mathbb{B}(i_2, j_1)$, $\mathbb{B}(i_1, j_2)$, and $\mathbb{B}(i_2, j_2)$ cannot be all 1's. Otherwise, one of the following three cases will happen.

Case 1: $|\{i_1, i_2\} \cap \{j_1, j_2\}|=0$: there would be a circuit $v_{i_1}v_{j_1}v_{i_2}v_{j_2}v_{i_1}$ of length four in G, contradicting $g(G) \geqslant 5$.

Case 2: $|\{i_1, i_2\} \cap \{j_1, j_2\}|=1$: say $i_1=j_1$ but $i_2 \neq j_2$, then there would be a triangle $v_{i_1}v_{i_2}v_{j_2}v_{i_1}$ in G, again contradicting $g(G) \geqslant 5$.

Case 3: $|\{i_1, i_2\} \cap \{j_1, j_2\}|=2$: then $(v_{i_1}, v_{i_2}) \in E$, a contradiction with the assumption that v_{i_1} is not adjacent to v_{i_2} in G.

So, the four entries, $\mathbb{B}(i_1, j_1)$, $\mathbb{B}(i_2, j_1)$, $\mathbb{B}(i_1, j_2)$, and $\mathbb{B}(i_2, j_2)$ are not all 1's, implying that making row i_1 and row i_2 adjacent in $\mathbb{B}$ makes at most one pair of 1's adjacent in all the columns.

We can conclude that making a pair of rows adjacent leads to at most two pairs of adjacent 1's in all the columns, and exactly two pairs of adjacent 1's if and only if the two vertices corresponding to the two rows are adjacent in G. Since in the circular case a row permutation involves exactly n pairs of adjacent rows, therefore it makes at most $2n$ pairs of 1's adjacent within all the columns, and the permutation can make $2n$ pairs of 1's adjacent in $\mathbb{B}$ if and only if the permutation represents a Hamiltonian circuit in G. But if there is a row permutation for $\mathbb{B}$ making each column having at most $k-1$ blocks, the permutation must make at least two pairs of 1's adjacent in each column (because each column has $k+1$ ones), and therefore it makes totally at least $2n$ pairs of 1's adjacent in $\mathbb{B}$, and thus there must be a Hamiltonian circuit in G for this permutation to represent. □

For the linear case, a row permutation of $\mathbb{B}$ involves exactly $n-1$ pairs of adjacent rows. A similar proof to that for Theorem 2.1 would prove the next theorem.

Theorem 2.2. *If $G=(V,E)$ is a graph such that $g(G)\geqslant 5$ and every vertex is of degree k except for two vertices a and b which are of degree $k-1$. Then G has a Hamiltonian path from a to b if and only if there is a row permutation for $\mathbb{A}(G)+\mathbb{I}$ resulting in a matrix having at most $k-1$ linear blocks per column.*

It is easy to see that the above proofs stay valid after replacing "at most" by "exactly". Therefore we have the following.

Theorem 2.3. *A k-regular graph $G=(V,E)$ with $g(G)\geqslant 5$ has a Hamiltonian circuit if and only if $\mathbb{A}(G)+\mathbb{I}$ can be row permuted to have exactly $k-1$ circular blocks of consecutive ones in each column.*

Theorem 2.4. *If $G=(V,E)$ is a graph such that $g(G)\geqslant 5$ and every vertex is of degree k except for two vertices a and b which are of degree $k-1$, then G has a Hamiltonian path from a to b if and only if $\mathbb{A}(G)+\mathbb{I}$ can be row permuted to have exactly $k-1$ linear blocks of consecutive ones in each column.*

3. The NP-completeness results

In this section, we prove that all of the following six problems are NP-complete, the first three of which are about linear blocks, and the last three circular blocks.

Two linear blocks for symmetric matrices (*2LBSM*):

Instance: A symmetric binary matrix $\mathbb{M}$ with no more than three linear blocks of consecutive ones in each row.

Question: Is there a row permutation for $\mathbb{M}$ that would result in a matrix having at most two linear blocks of consecutive ones per column?

Linear blocks for 3-block-row matrices (*LB3M*):

Instance: A binary matrix $\mathbb{M}$ with no more than three linear blocks of consecutive ones per row; and fixed constant $k\geqslant 2$.

Question: Is there a row permutation for $\mathbb{M}$ that would result in a matrix having at most k linear blocks of consecutive ones per column?

Linear blocks for symmetric matrices (*LBSM*):

Instance: A symmetric binary matrix $\mathbb{M}$; and fixed constant $k\geqslant 2$.

Question: Is there a row permutation for $\mathbb{M}$ that would result in a matrix having at most k linear blocks of consecutive ones per column?

Two circular blocks for symmetric matrices (*2CBSM*):

Instance: A symmetric binary matrix $\mathbb{M}$ with no more than three circular blocks of consecutive ones in each row.

Question: Is there a row permutation for $\mathbb{M}$ that would result in a matrix having at most two circular blocks of consecutive ones per column?

Circular blocks for 3-block-row matrices (*CB3M*):

Instance: A binary matrix $\mathbb{M}$ with no more than three circular blocks of consecutive ones per row; and fixed constant $k\geqslant 2$.

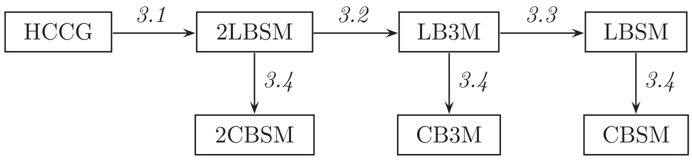

Fig. 1. Transformations among the seven problems.

Question: Is there a row permutation for $\mathbb{M}$ that would result in a matrix having at most k circular blocks of consecutive ones per column?

Circular blocks for symmetric matrices (*CBSM*):

Instance: A symmetric binary matrix $\mathbb{M}$; and fixed constant $k \geqslant 2$.

Question: Is there a row permutation for $\mathbb{M}$ that would result in a matrix having at most k circular blocks of consecutive ones per column?

The starting point of our transformation is the following NP-complete problem [5].

Hamiltonian circuit in cubic graphs (*HCCG*):

Instance: A bridgeless cubic graph $G = (V, E)$.[1]

Question: Does G contains a Hamiltonian circuit?

In fact, the result given in [5] is much stronger: they proved that the well-known Hamiltonian Circuit Problem remains NP-complete when restricted to graphs that are cubic, 3-connected, planar, and have no face with fewer than five edges. Fig. 1 shows the polynomial transformations among the seven problems. The numbers with the arrows are the subsections where we present the transformations.

3.1. Two linear blocks for symmetric matrices (2LBSM)

This section proves the NP-completeness of 2LBSM by a polynomial transformation from HCCG. The transformation needs to use a *matching* in a constructed graph.[2] The existence of the matching is guaranteed by the following lemma.

Lemma 3.1. *Every bridgeless graph* $G = (V, E)$ *with two specific vertices* $a, b \in V$ *such that* $d(a) = d(b) = 2$ *and* $d(v) = 3$ *for every vertex* v *other than* a *and* b *has a matching*.

Proof. Petersen's theorem of [11, Corollary 2.2.3] states that every bridgeless cubic graph has a matching. Here, the graph G is bridgeless but not strictly cubic because of the two specific vertices a and b which are of degree 2. Fortunately, the existence proof of matching in this situation can be conducted similarly as that for Petersen's theorem in [11].

We show that the graph $G = (V, E)$ satisfies Tutte's condition: $\forall S \subseteq V$, $q(G - S) \leqslant |S|$, where $q(G - S)$ is the number of odd components of $G - S$.[3] Tutte's condition is necessary and sufficient for a graph to have a matching [11]. Let $S \subseteq V$ be given, and consider an odd component C of $G - S$ which has no specific vertices (a or b). The degrees of the vertices in C sum to an odd number, $\sum_{v \in V(C)} d(v) = 3|C|$, but the edges of C, $E(C)$, contribute only an even number, $2|E(C)|$, to the sum (each edge contributes two); the difference between the sum and the contribution, $3|C| - 2|E(C)|$, is odd and due to the edges between S and C. So G has an odd number of edges crossing S and C, and therefore has at least three such edges (since G has no bridge). There are at least $q(G - S) - 2$ odd components involving no specific vertices (since there are only two specific vertices), and every other component C has at least two edges crossing S and C (again, because G has no bridge). The total number of edges crossing S and $G - S$ is at least $3(q(G - S) - 2) + 4$. All of them are the incident edges of the vertices of S and each vertex of S is incident upon at most three of them. So, $3|S| \geqslant 3(q(G - S) - 2) + 4$, implying Tutte's condition. □

Theorem 3.1. 2*LBSM is* NP-*complete*.

Proof. Starting from any bridgeless cubic graph G, an instance of HCCG, we present a way to construct in polynomial time a symmetric binary matrix $\mathbb{M}$ with each row having no more than three linear blocks such that $\mathbb{M}$ can be row

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[1] A *bridge* is an edge of G such that $G - e$ is not connected; *cubic* is synonymous with 3-*regular*.

[2] A matching M in G is a set of independent edges of G such that each vertex is incident on exactly one edge of M.

[3] An odd component is a maximally connected subgraph with an odd number of vertices.

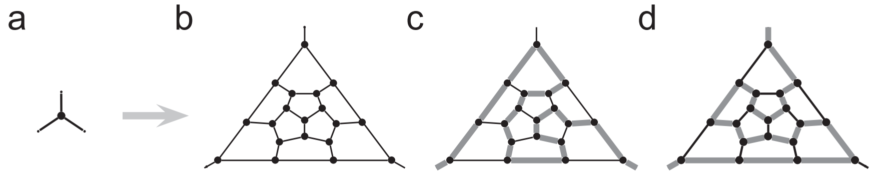

Fig. 2. The replaced vertex (a) and the substitution graph (b) with possible local paths (c) and (d) (alternate and symmetric paths omitted). A Hamiltonian circuit, once entering the substitution graph, must visit all 19 vertices before leaving the substitution graph.

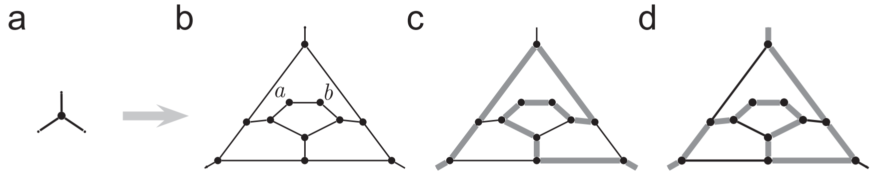

Fig. 3. The replaced vertex (a) and the substitution graph (b) with possible local paths (c) and (d) (alternate and symmetric paths omitted). A Hamiltonian circuit, once entering the substitution graph, must pass through the edge (a,b) and visit all 11 vertices before leaving the substitution graph.

permuted to make each column having at most two linear blocks if and only if G has a Hamiltonian circuit. The construction proceeds in four steps.

The first step is to make the girth of the graph to be at least 5 while keeping the bridgeless and cubic properties and the Hamiltonicity. This is achieved by replacing each vertex of G with the substitution graph shown in Fig. 2 (due to [5]). The resulted graph G' obviously remains bridgeless and cubic and has a girth $g(G') \geqslant 5$. Moreover, G' and G concurrently have or have no Hamiltonian circuits.

The second step is to choose an arbitrary vertex v in G' to be replaced by the substitution graph shown in Fig. 3. Obviously, the resulted graph G'' is bridgeless and $g(G'') \geqslant 5$. It is cubic except for the two special vertices a and b. Moreover, because every Hamiltonian circuit (if any) in G'' must pass through the edge (a, b), G'' has a Hamiltonian path from a to b if and only if G'' has a Hamiltonian circuit, if and only if G' has a Hamiltonian circuit, and if and only if G has a Hamiltonian circuit.

The third step is using G'' to construct matrix $\mathbb{B} = \mathbb{A}(G'') + \mathbb{I}$. $\mathbb{B}$ is a symmetric binary matrix. By Theorem 2.2, $\mathbb{B}$ can be row permuted to have at most two linear blocks per column if and only G'' contains a Hamiltonian path from a to b, and if and only if G contains a Hamiltonian circuit.

Clearly, the above three steps can be completed within polynomial time. The last step is to permute the rows and the columns of $\mathbb{B}$ to result in a symmetric matrix $\mathbb{M}$ that has at most three linear blocks per row—an instance of 2LBSM. This can be done in polynomial time, as follows.

Lemma 3.1 guarantees that the graph G'' has a matching M. The matching M can be found in polynomial time using standard techniques (see, for example, [10]). Consequently, we can for each $(v_i, v_j) \in M$ make column i and column j adjacent to let the two 1-entries $\mathbb{B}(i, i)$ and $\mathbb{B}(i, j)$ be adjacent in row i and the two 1-entries $\mathbb{B}(j, i)$ and $\mathbb{B}(j, j)$ be adjacent row j; at the same time, we can make row i and row j adjacent to keep the symmetry of the matrix, giving rise to a symmetric matrix $\mathbb{M}$ that has at most three blocks in each row (because each row of $\mathbb{B}$ has at most four 1's). This ends the proof. □

3.2. Linear blocks for 3-block-row matrices (LB3M)

For every constant $k \geqslant 3$, our NP-completeness proof for LB3M employs a matrix that depends only on k and is constructed through Lemmas 3.2 and 3.3.

Lemma 3.2. *For any $k \geqslant 3$, we can construct a matrix $\mathbb{Q}$ of size $4(k-2)^2 \times 4(k-2)$ such that*

(i) *each row of $\mathbb{Q}$ has exactly two* 1*'s*;
(ii) *by any row permutation, there is a column having at least $k-1$ blocks*;
(iii) *there is a row permutation by which $\mathbb{Q}$ has at most $k-1$ blocks per column.*

Proof. Let $\mathbb{I}$ be the $2(k-2)\times 2(k-2)$ unit matrix, and $\mathbf{1}^{\mathrm{T}}$ be a $2(k-2)\times 1$ matrix with all 1-entries. Then $\mathbb{Q}$ is obtained by vertically juxtaposing $2(k-2)$ copies of $\mathbb{I}$ and then horizontally appending a matrix obtained by diagonally duplicating $\mathbf{1}^{\mathrm{T}}$ for $2(k-2)$ times, as shown below.

$$\mathbb{Q}=\begin{pmatrix} \mathbb{I} & \mathbf{1}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \dots & \mathbf{0}^{\mathrm{T}} \\ \mathbb{I} & \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}} & \dots & \mathbf{0}^{\mathrm{T}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{I} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \dots & \mathbf{1}^{\mathrm{T}} \end{pmatrix},\quad \text{where } \mathbf{1}^{\mathrm{T}}=\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \quad\text{and}\quad \mathbf{0}^{\mathrm{T}}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}.$$

Clearly, $\mathbb{Q}$ has $4(k-2)^2$ rows and $4(k-2)$ columns, with exactly two ones per row and $2(k-2)$ ones per column. (i) is proved.

Note that any 2×2 submatrix of $\mathbb{Q}$ cannot contain all ones, which means that making any two rows of Q adjacent makes at most one pair of 1's adjacent in all the $2(k-2)$ columns. So, any row permutation of $\mathbb{Q}$ makes at most $4(k-2)^2-1$ pairs of 1's adjacent in all the columns because it makes only $4(k-2)^2-1$ pairs of rows adjacent in the linear case. Hence, by any row permutation there must exist a column of $\mathbb{Q}$ with fewer than $k-2$ pairs of adjacent 1's, implying that the column has at least $k-1$ blocks (because there are $2(k-2)$ ones in the column). We have (ii).

To show (iii), we present a permutation by which each column has no more than $k-1$ blocks. We use $q_{i,j}$ to denote the jth row of the ith $\mathbb{I}$ in $\mathbb{Q}$ (the $(2(k-2)(i-1)+j)$th row of $\mathbb{Q}$). The permutation achieving at most $k-1$ blocks per column arranges the rows in the following order:

$$\text{For } j=1,3,5,\dots,2(k-2)-1: q_{1,j}q_{1,j+1}q_{2,j+1}q_{2,j}q_{3,j}q_{3,j+1}q_{4,j+1}q_{4,j}\cdots$$
$$q_{2(k-2)-1,j}q_{2(k-2)-1,j+1}q_{2(k-2),j+1}q_{2(k-2),j}.$$

We skip the straightforward verification that there are indeed $k-1$ blocks in each of the columns $1,3,\dots,2(k-2)-1$, and $k-2$ blocks in each of the other columns. The verification implies (iii). □

Lemma 3.3. *For any $k\geqslant 3$, we can construct a matrix $\mathbb{R}$ of size*

$$(40(k-2)^3+2k-3)\times 40(k-2)^2$$

such that

(a) *each row of $\mathbb{R}$ has no more than two blocks*;
(b) *by any row permutation, there is a column having at least k blocks*;
(c) *there is a row permutation by which $\mathbb{R}$ has at most k blocks per column*;
(d) *in $\mathbb{R}$ there are $2k-3$ rows $\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_{2k-3}$ called r-rows in the following such that any optimal permutation (which achieves no more than k blocks per column) must permute them consecutively in the order of $\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_{2k-3}$, or the reverse*; *and*
(e) *each of the $2k-3$ r-rows contains exactly one block.*

Proof. By diagonally duplicating $\mathbb{Q}$ of Lemma 3.2 for 5 times we get

$$\mathbb{Q}_5=\begin{pmatrix} \mathbb{Q} & & & & \\ & \mathbb{Q} & & & \\ & & \mathbb{Q} & & \\ & & & \mathbb{Q} & \\ & & & & \mathbb{Q} \end{pmatrix},$$

a matrix of size $20(k-2)^2 \times 20(k-2)$ having the properties (i)–(iii) of Lemma 3.2. Note that, if two rows r_1, r_2 with all 1-entries are added to $\mathbb{Q}_5$ to form

$$\mathbb{T} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbb{Q}_5 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbb{Q} & & & & \\ & \mathbb{Q} & & & \\ & & \mathbb{Q} & & \\ & & & \mathbb{Q} & \\ & & & & \mathbb{Q} \end{pmatrix},$$

then for this matrix $\mathbb{T}$, we can let row r_1 be adjacent to row r_2 and extend this adjacency to a row permutation by which each column would have at most k blocks. On the other hand, if r_1 and r_2 are not adjacent, since each of them can be adjacent to at most two rows in any row permutation, among the five $\mathbb{Q}$'s, there must be one $\mathbb{Q}$ that has no rows adjacent to r_1 or r_2; thus, there is some column of $\mathbb{T}$ involving that $\mathbb{Q}$ having at least $(k-1)+2=k+1$ blocks; so, r_1 and r_2 must be adjacent if no more than k blocks can be in each column.

With the above observation on $\mathbb{T}$, the required matrix $\mathbb{R}$ is constructed as follows:

$$\mathbb{R} = \begin{pmatrix} \mathbf{1} & & & \\ \mathbf{1} & \mathbf{1} & & \\ & \mathbf{1} & \ddots & \\ & & \ddots & \mathbf{1} \\ & & & \mathbf{1} \\ \mathbb{Q}_5 & & & \\ & \mathbb{Q}_5 & & \\ & & \ddots & \\ & & & \mathbb{Q}_5 \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \vdots \\ \mathbf{r}_{2k-4} \\ \mathbf{r}_{2k-3} \\ \hline \mathbb{S} \end{pmatrix}.$$

Roughly, $\mathbb{R}$ is obtained by first duplicating $\mathbb{Q}_5$ along the diagonal for $2k-4$ times to form matrix $\mathbb{S}$, and then appending $2k-3$ r-rows $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{\mathbf{2k-3}}$ (on top of $\mathbb{S}$), such that all the columns of $\mathbb{R}$ involving the ith $\mathbb{Q}_5$ form a submatrix that (by the observation on $\mathbb{T}$) forces row $\mathbf{r}_i$ to be adjacent with row $\mathbf{r}_{i+1}$. It can be seen that $\mathbb{R}$ has $40(k-2)^3+2k-3$ rows and $40(k-2)^2$ columns and satisfies (a)–(e), thus completing the proof. □

Note that the R stated in Lemma 3.3 is constructed in constant time if k is a constant.

Theorem 3.2. *LB3M is* NP-*complete for every fixed* $k \geqslant 2$.

Proof. If $k=2$, the theorem is implied by Theorem 3.1. So we can assume $k \geqslant 3$. We transform 2LBSM to LB3M. Let $\mathbb{M}_{n\times n}$ be any instance of 2LBSM and $\mathbb{R}$ be the matrix stated in Lemma 3.3; then the matrix for LB3M is

$$\mathbb{M}' = \left(\begin{array}{c|c} \mathbb{M} & \mathbb{O} \\ \hline \mathbf{0} & \mathbf{r}_1 \\ \mathbf{1} & \mathbf{r}_2 \\ \mathbf{0} & \mathbf{r}_3 \\ \vdots & \vdots \\ \mathbf{1} & \mathbf{r}_{2k-4} \\ \mathbf{0} & \mathbf{r}_{2k-3} \\ \hline \mathbb{O} & \mathbb{S} \end{array}\right),$$

of which the two $\mathbb{O}$'s are matrices of appropriate sizes with all 0-entries. That is, $\mathbb{M}$ is obtained by (1) joining $\mathbb{M}$ and $\mathbb{R}$ in the main diagonal (where $\mathbb{M}$ is a submatrix of $\mathbb{M}'$ covering the first n rows and the first n columns of $\mathbb{M}'$), (2) assigning 1 to the entries underneath $\mathbb{M}$ (i.e., in the first n columns) and corresponding to the $k-2$ r-rows $\mathbf{r}_2, \mathbf{r}_4, \ldots, \mathbf{r}_{\mathbf{2(k-2)}}$ of

$\mathbb{R}$, and (3) leaving other entries that are neither in $\mathbb{M}$ nor in $\mathbb{R}$ to be 0's. Clearly, for constant k the matrix $\mathbb{M}'$ can be constructed within time polynomial of n.

$\mathbb{M}'$ has no more than three blocks per row, because $\mathbb{R}$ has no more than two blocks per row and $\mathbb{M}$ has no more than three blocks per row. So $\mathbb{M}'$ and k form an instance of LB3M.

Note that underneath $\mathbb{M}$ each of the n columns has exactly $k-2$ ones. If $\mathbb{M}$ can be row permuted such that each column has at most two blocks, by (c) of Lemma 3.3, the row permutation for $\mathbb{M}$ can be extended to the rows of $\mathbb{M}'$ with which each column of $\mathbb{M}'$ has at most k blocks.

Conversely, if there exists a row permutation for $\mathbb{M}'$ leading to each column having no more than k blocks, then by (d) of Lemma 3.3, this permutation must arrange the $2k-3$ r-rows in the order (or the reverse order) of $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{\mathbf{2k-3}}$; this order separates the $k-2$ r-rows $\mathbf{r}_2, \mathbf{r}_4, \ldots, \mathbf{r}_{\mathbf{2(k-2)}}$ from one another, and thus $k-2$ blocks are formed in each of the n columns underneath $\mathbb{M}$. So the permutation must make the matrix $\mathbb{M}$ to have at most two blocks per column. In conclusion, $\mathbb{M}'$ can be row permuted to have at most k blocks per column if and only if $\mathbb{M}$ can be row permuted to have at most two blocks per column. □

3.3. *Linear blocks for symmetric matrices (LBSM)*

Theorem 3.3. *LBSM is* NP-*complete for every fixed* $k \geqslant 2$.

Proof. The case of $k=2$ is directly implied by Theorem 3.1. We assume $k \geqslant 3$. For a given matrix $\mathbb{M}_{m\times n}$ with no more than three blocks per row, we construct a symmetric matrix $\mathbb{M}'$ such that LB3M replies "yes" for $\mathbb{M}$ and k if and only if LBSM replies "yes" for $\mathbb{M}'$ and k. Such an $\mathbb{M}'$ can be simply constructed by joining $\mathbb{M}$ and $\mathbb{M}^{\mathrm{T}}$ (the transpose of $\mathbb{M}$) in the secondary diagonal, that is,

$$\mathbb{M}' = \begin{pmatrix} \mathbb{O} & \mathbb{M} \\ \mathbb{M}^{\mathrm{T}} & \mathbb{O} \end{pmatrix},$$

where $\mathbb{O}$ is a matrix of the appropriate size with all 0-entries. Obviously, (1) $\mathbb{M}'$ is an $(m+n)\times(m+n)$ symmetric matrix computable from $\mathbb{M}$ in polynomial time, and (2) for $k \geqslant 3$, $\mathbb{M}'$ can be row permuted so that each column has at most k blocks if and only if $\mathbb{M}$ can be row permuted to achieve the same. Thus the construction of $\mathbb{M}'$ from $\mathbb{M}$ is a polynomial transformation from LB3M to LBSM. The NP-completeness of LBSM follows from that of LB3M. □

3.4. *Circular blocks*

Theorems 3.1–3.3 are for the case of linear blocks. In fact, they also hold for the case of circular blocks. The transformations are simple. To a matrix $\mathbb{M}$, an instance of 2LBSM or LB3M or LBSM, simply append a row and a column of all 0-entries, giving us

$$\mathbb{M}' = \begin{pmatrix} \mathbb{M} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & 0 \end{pmatrix}.$$

Then, $\mathbb{M}'$ is symmetric if and only if $\mathbb{M}$ is symmetric, $\mathbb{M}'$ has no more than 3 circular blocks per row if and only if $\mathbb{M}$ has no more than three linear blocks per row, and $\mathbb{M}'$ can be row permuted to have at most k circular blocks per column if and only if $\mathbb{M}$ can be row permuted to have at most k linear blocks per column. So we have the following.

Theorem 3.4. 2*CBSM is* NP-*complete*, *and for every fixed* $k \geqslant 2$, *CB3M and CBSM are* NP-*complete*.

3.5. *Remarks*

We pointed out that 2LBSM and 2CBSM remain NP-complete after replacing "at most" in their questions with "exactly". This directly implies the NP-completeness result by Kou [9]—determining whether a given matrix can be row permuted to have at most k blocks in all columns is NP-complete.

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