Comments

Comments on "A New Family of Cayley Graph Interconnection Networks of Constant Degree Four"

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Abstract—Vadapalli and Srimani [2] have proposed a new family of Cayley graph interconnection networks of constant degree four. Our comments show that their proposed graph is not new but is the same as the wrap-around butterfly graph. The structural kinship of the proposed graph with the de Bruijn graph is also discussed.

Index Terms— Interconnection network, Cayley graph, generator, de Bruijn graph, butterfly graph, isomorphism.

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1 Definition of Graph $\mathcal{G}(n)$

WE first give the definition of the graph $\mathcal{G}(n)$ proposed by Vadapalli and Srimani [2].

Each node of $\mathcal{G}(n)$ is represented as a circular permutation of n different symbols in lexicographic order, where the n symbols are presented in either uncomplemented or complemented form. Let t_k, $0 \le k \le n-1$, denote the kth symbol in the set of n symbols. We use the English alphabet for the symbols: thus, for $n = 4$, $t_0 = a$, $t_1 = b$, $t_2 = c$, and $t_3 = d$. We use t_k^* to denote either t_k or $\bar{t}_k$. Therefore, for n distinct symbols, there are exactly n different cyclic permutations of the symbols in lexicographic order, and, since each symbol can be present in either uncomplemented or complemented form, the node set of $\mathcal{G}(n)$ has a cardinality of $n \times 2^n$. Since each node is some cyclic permutation of the n symbols in lexicographic order, then, if $a_0a_1\ldots a_{n-1}$ denotes the label of an arbitrary node and $a_0 = t_k^*$ for some integer k, then, for all i, $1 \le i \le n-1$, we have $a_i = t^*_{(k+i)\ (\mathrm{mod}\ n)}$. Thus, the definition of $\mathcal{G}(n)$ is given as follows.

DEFINITION 1. *The graph $\mathcal{G}(n)$ is a Cayley graph whose nodes comprise the $n \times 2^n$ cyclic permutations of n distinct symbols in lexicographic order. Each symbol is presented in either uncomplemented or complemented form. Given a node represented as a string $a_0a_1\ldots a_{n-1}$, its edges are defined by the following generators*:

$$g(a_0a_1\ldots a_{n-1}) = a_1a_2\ldots a_{n-1}a_0$$

$$f(a_0a_1\ldots a_{n-1}) = a_1a_2\ldots a_{n-1}\bar{a}_0$$

$$g^{-1}(a_0a_1\ldots a_{n-1}) = a_{n-1}a_0\ldots a_{n-2}$$

$$f^{-1}(a_0a_1\ldots a_{n-1}) = \bar{a}_{n-1}a_0\ldots a_{n-2}\,.$$

If the identity permutation is $t_0t_1\ldots t_{n-1}$, then the generator set $\Omega = \{f, g, f^{-1}, g^{-1}\}$ is given as:

$$g = t_1t_2\ldots t_{n-1}t_0$$

$$f = t_1t_2\ldots t_{n-1}\bar{t}_0$$

$$g^{-1} = t_{n-1}t_0\ldots t_{n-2}$$

$$f^{-1} = \bar{t}_{n-1}t_0\ldots t_{n-2}\,.$$

Fig. 1a shows $\mathcal{G}(3)$ drawn in a "regular" fashion, which is different from that in [2]. The identity permutation of $\mathcal{G}(3)$ is abc, and the generator set is $\{bca, bc\bar{a}, cab, \bar{c}ab\}$. The nodes of $\mathcal{G}(n)$ are grouped into different columns according to the position of the first symbol t_0^* in their labels. In Fig. 1a, nodes with the symbol a in the leftmost position of their labels form the first column, nodes with the symbol a in the rightmost position form the second column, and nodes with the symbol a in the middle position form the third column. The first column is duplicated in order to give a clearer view of the connections. We use solid lines to denote the g-edges, i.e., the edges defined by the permutation g or g^{-1}, and dotted lines to denote the f-edges.

2 Isomorphism to the Wrap-Around Butterfly Graph

In this section, we prove that the graph $\mathcal{G}(n)$ is isomorphic to the wrap-around butterfly graph $\mathcal{B}(n)$.

DEFINITION 2. *The wrap-around butterfly graph $\mathcal{B}(n)$ has node-set $Z_n \times Z_2^n$. Each node is represented as a pair $\langle c, r\rangle$, where $c \in Z_n$ is the column of the node and $r \in Z_2^n$ is the row of the node. The edges of $\mathcal{B}(n)$ form butterflies (i.e., copies of the complete bipartite graph $\mathcal{K}_{2,2}$) between consecutive columns of nodes. Each node $\langle c, r\rangle$ is connected to the node $\langle c', r\rangle$ and the node $\langle c', r'\rangle$, where $c' = c + 1 \ (\mathrm{mod}\ n)$ and r' and r differ in precisely the cth bit; the first edge is a straight edge and the second edge is a cross edge.*

Fig. 1c shows $\mathcal{B}(3)$.

An isomorphical mapping between $\mathcal{G}(n)$ and $\mathcal{B}(n)$ is as follows: Given an arbitrary node $a_0a_1\ldots a_{n-1}$ in $\mathcal{G}(n)$ and $a_k = t_0^*$ for some k, the node a becomes $a' = a_ka_{k+1}\ldots a_{n-1}a_0\ldots a_{k-1}$ after $(n-k)$ $(\mathrm{mod}\ n)$ g^{-1} operations. If we substitute a 0 for every uncomplemented symbol and a 1 for every complemented symbol in a', and let the resulting binary string be r, then node $a_0a_1\ldots a_{n-1}$ in $\mathcal{G}(n)$ corresponds to node $\langle n-k\ (\mathrm{mod}\ n), r\rangle$ in $\mathcal{B}(n)$. It is not difficult to see that this mapping is a bijection. Furthermore, the g-edges in $\mathcal{G}(n)$ correspond to the direct edges in $\mathcal{B}(n)$, while the f-edges in $\mathcal{G}(n)$ correspond to the cross edges of $\mathcal{B}(n)$. To see the latter, consider nodes $a = a_0a_1\ldots a_{n-1}$ and $b = a_1\ldots a_{n-1}a_0$ in $\mathcal{G}(n)$. a and b are connected by a g-edge. According to the above mapping, a corresponds to the node $\langle n-k\ (\mathrm{mod}\ n), r\rangle$ in $\mathcal{B}(n)$, where $n-k$ and r are computed as in the above; since $b = g(a)$, $t_k^* = a_k$ is at position $k-1 \ (\mathrm{mod}\ n)$ in b, and, so, b corresponds to the node $\langle n-k+1\ (\mathrm{mod}\ n), r\rangle$; clearly, these two nodes in $\mathcal{B}(n)$ are connected by a direct edge, by the definition of $\mathcal{B}(n)$. A similar analysis can be applied to the mapping between an f-edge in $\mathcal{G}(n)$ and a cross edge in $\mathcal{B}(n)$.

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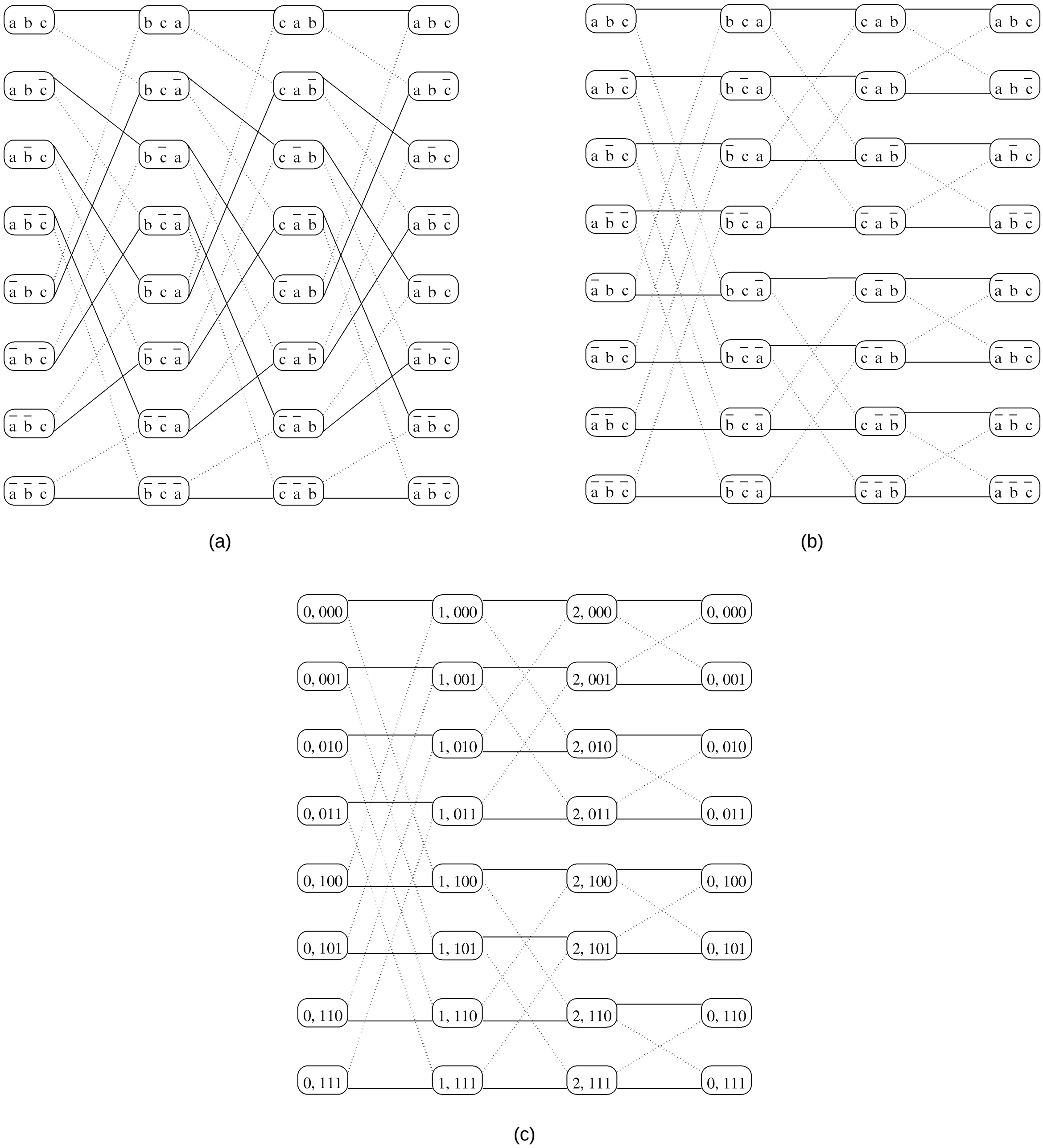

Fig. 1. (a) The proposed network, (b) after "straightening" the *g*-edges, (c) a wrap-around butterfly network. The solid lines are the *g*-edges in (a), (b), or the straight edges in (c); the dotted lines are the *f*-edges in (a), (b), or the cross edges in (c).

Refer again to Fig. 1 for an example. Based on the fact that a *g*-edge in $\mathcal{G}(n)$ corresponds to a direct edge in $\mathcal{B}(n)$, we "straighten" all the *g*-edges in $\mathcal{G}(3)$ (Fig. 1a) (thus reordering the nodes in each column), and the result is the $\mathcal{G}(3)$, as shown in Fig. 1b. Clearly, the latter is the same as the $\mathcal{B}(n)$ in Fig. 1c.

3 FURTHER DISCUSSION

We have shown that the graph $\mathcal{G}(n)$ proposed by Vadapalli and Srimani is not a new graph, but a new representation of the wrap-around butterfly graph. Indeed, $\mathcal{G}(n) = \mathcal{B}(n)$.

The group-theoretic relations between $\mathcal{B}(n)$ (or $\mathcal{G}(n)$) and the de Bruijn graph are well studied in [1], where $\mathcal{B}(n)$ is proved to be a Cayley graph derived from the de Bruijn graph acting as a group action graph, and, inversely, the de Bruijn graph is proved to be some coset graph of $\mathcal{B}(n)$.

The new representation in [2] shows another simple structural kinship between $\mathcal{G}(n)$ (or $\mathcal{B}(n)$) and the de Bruijn graph. In particular, if n distinct symbols in $\mathcal{G}(n)$ are the same, i.e., each bit of the node address of $\mathcal{G}(n)$ is either 0 or 1, $\mathcal{G}(n)$ specializes to become the de Bruijn graph.

The new representation in [2] may bring about some convenience in studying the topological properties of $\mathcal{G}(n)$ (or $\mathcal{B}(n)$), such as optimal routing algorithms and fault tolerance.

REFERENCES

[1] F. Annexstein, M. Baumslag and A.L. Rosenberg, "Group Action Graphs and Parallel Architectures," *SIAM J. Computing*, vol. 19, no. 3, pp. 544–569, June 1990.

[2] P. Vadapalli and P.K. Srimani, "A New Family of Cayley Graph Interconnection Networks of Constant Degree Four," *IEEE Trans. Parallel and Distributed Systems*, vol. 7, no. 1, pp. 26–32, Jan. 1996.